UNCLASSIFIED

AD NUMBER
AD488395
LIMITATION CHANGES
TO: **Approved for public release; distribution is unlimited.**
FROM: **Distribution authorized to U.S. Gov't. agencies and their contractors; Administrative/Operational Use; JUN 1966. Other requests shall be referred to Office of Naval Research, Arlington, VA 22203.**
AUTHORITY
ONR ltr 27 Jul 1971

THIS PAGE IS UNCLASSIFIED

COLUMBIA UNIVERSITY

Technical Report No. 94

REPRESENTATION AND REALIZATION OF TIME-VARIABLE LINEAR SYSTEMS

by

Leonard M. Silverman

Prepared for

DEPARTMENT OF THE NAVY
Office of Naval Research
Mathematics Branch
Washington, D. C. 20360

Contract NONR4259(04)

NR 044-311

June, 1966

DEPARTMENT OF ELECTRICAL ENGINEERING
SCHOOL OF ENGINEERING AND APPLIED SCIENCE
NEW YORK 27, N.Y.

PRECIS

Technical Report

Title: "Representation and Realization of Time-Variable Linear Systems", Leonard M. Silverman, Department of Electrical Engineering Technical Report #94, Columbia University; NONR 4259(04).

Background: This report is an outgrowth of a study of nonlinear and time-variable systems being conducted in the Department of Electrical Engineering, Columbia University. Contained in the report are the results of an investigation of the subclass of such systems described by a system of first-order linear time-variable differential equations.

Condensed Report Contents: Systems of linear time-variable differential equations are studied, with particular emphasis on identifying those system properties and concepts that can be characterized without knowledge of the equation's solution. Criteria are developed for determining the degree of controllability and observability of such systems. These criteria are based on the rank of matrices formed directly from the system coefficients. A transformational property of these matrices is utilized in a procedure for reducing noncontrollable and nonobservable systems to lower dynamic order. Also obtained are new methods for characterizing and generalizing equivalent system representations, including criteria for equivalence and zero-state time invariance of time-variable systems.

Based on the theory of equivalent systems, a new approach to the synthesis of nonstationary impulse response matrices is developed. This method, which does not require an a priori assumption of separability, provides a systematic procedure for realizing a wide class of responses.

Application is also made to time-variable electrical networks. By relating the concepts of controllability and unilateral transmission, the existences of a class of unilateral networks composed solely of two-terminal RC (time-variable) components is established.

For Further Information: The complete report is available from the Defense Documentation Center for qualified requestors.

COLUMBIA UNIVERSITY

Technical Report No. 94

REPRESENTATION AND REALIZATION OF TIME-VARIABLE LINEAR SYSTEMS

by

Leonard M. Silverman

Prepared for

DEPARTMENT OF THE NAVY
Office of Naval Research
Mathematics Branch
Washington, D. C. 20360

Contract NONR4259(04)

NR 044-311

June, 1966

DEPARTMENT OF ELECTRICAL ENGINEERING
SCHOOL OF ENGINEERING AND APPLIED SCIENCE
NEW YORK 27, N. Y.

ABSTRACT

This research is concerned with a study of time-variable linear systems with particular emphasis on those system properties and concepts that can be characterized without solving time-variable differential equations. Criteria are developed for determining the degree of controllability and observability of a time-variable system. These criteria are based on the rank of the controllability and observability matrices, newly defined quantities formed from the system coefficient matrices and a finite number of their derivatives. The controllability and observability matrices are also shown to be useful in a variety of other system analysis problems. In particular, a transformational property of these matrices is utilized in an explicit method for reducing noncontrollable and nonobservable systems to systems of lower dynamic order. Also obtained are new methods for characterizing and generating equivalent system representations, including criteria for equivalence, zero-state equivalence and zero state time-invariance of time-variable systems. Based on the theory of equivalent systems and the criteria for controllability and observability, a new approach to the synthesis of nonstationary impulse response matrices is developed. This method of synthesis, which does not require an a priori assumption of separability, provides a systematic procedure

for realizing a wide class of impulse responses. By relating the concepts of controllability and unilateral transmission, the existence of a class of unilateral networks composed solely of two-terminal RC (time-variable) components is demonstrated. A stable example of such a network is presented and possible applications discussed.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Motivation

In recent years there has been a growing interest in the theory and application of systems whose parameters vary with time. Among the types of systems for which this is the case are electrical networks, control systems and communication channels. Some examples of inherently time-variable networks include parametric amplifiers, modulators and switched networks. In addition, problems involving control of missile flights, or communication over fading channels often may be analyzed by techniques applicable to time-variable systems. A fairly general survey of research on time-variable systems in the period 1950-1960 may be found in the paper of Zadeh [1]. For contributions prior to 1950, the paper of Bennett [2] is valuable. A great deal of emphasis in the past five or six years has been on those systems (both fixed and time-variable) that can be adequately represented in state-variable form [3]. This has especially been true in modern control theory (see the review paper of Athans [4]), and more recently in network analysis [5-11].

The major advantage of the state-variable approach to the analysis of time-variable systems is that an essentially geometric structure is given to the differential equations which describe system behavior. This structure considerably reduces notational complexity

and furthers insight into many analysis and synthesis problems.

Although there is a fairly complete theory of time-variable systems in state-variable form [3] this theory is for the most part based on an explicit knowledge of system solutions. It is generally impossible, however, to find the solution to a set of time-variable differential equations in closed form. While it is possible to compute the solution to any desired degree of accuracy on a digital computer, it cannot be characterized directly in terms of the system parameters. This is one reason why relatively little progress has been made in the analysis, synthesis and application of variable parameter systems compared with what has been accomplished for fixed systems.

These considerations motivate a closer examination of several important properties of time-variable systems with the intent of characterizing them as completely as possible in terms of ' own system parameters, and exploiting them in several specific applications.

1.2 Problem Formulation and Background

The major system concepts examined here are described below. Precise definitions are given in the body of the thesis.

(1) The concept of controllability is essentially concerned with the type of coupling that exists between input and state of a linear system, and determines the extent to which a system can be controlled. Dual to controllability, the concept

of observability is concerned with the type of coupling that exists between state and output of a linear system, and determines the extent to which a system's behavior can be observed.

(2) Related to controllability and observability is the idea of system reducibility. If a system's input-output behavior for zero-initial conditions can be achieved with a system of lower dynamic order, it is said to be reducible. Noncontrollable or nonobservable systems are always reducible.

(3) To meaningfully discuss analysis and synthesis, some notion of system equivalence is essential. Two systems whose states are related by a nonsingular transformation of coordinates (possibly time-variable) are said to be equivalent. Also important is the concept of zero-state equivalence. Two systems having the same input-output response for zero initial conditions are said to be zero-state equivalent. Thus, a reduced system is zero-state equivalent to its original description.

The most fundamental of the above system properties are controllability and observability. It has become increasingly evident in recent years that these concepts are significant in diverse problems of system analysis and synthesis not necessarily related to control problems. However, controllability first played an important role in the development of optimal control theory. Criteria which may be viewed

as types of controllability were present, for example in the work of Pontryagin [12], Gamkrelidze [13], LaSalle [14], and Bertram and Sarachik [15]. Formal definition of controllability and its dual observability was made by Kalman [16-18], who also showed the importance of observability in optimal filtering problems [19, 20]. Kreindler and Sarachik [21] introduced the concept of output controllability, and indicated some subtleties differentiating various degrees of controllability and observability in the time-variable case. Unfortunately, the necessary and sufficient conditions formulated by Kalman and his colleagues [18, 19, 20, 22, 23], and by Kreindler and Sarachik [21], for the various forms of controllability and observability depend explicitly on the system solution in the time-variable case. As this is generally not available except as a numerically tabulated solution, it has been difficult to analytically characterize the controllability and observability properties of a time-variable system. The same is true for reducibility and for system equivalence, since known conditions for these properties are also given in terms of system solutions.

It is our intent to obtain criteria for controllability, observability and system equivalence which in no way depend on knowledge of solutions to time-variable differential equations. In addition, methods for constructing zero-state equivalents of reducible systems and for finding explicit transformations between equivalent systems will be examined.

Several specific problems, whose solutions depend on an explicit characterization of the above properties, will also be attacked. These include:

(1) Conditions for the existence of simple canonical forms for time-variable systems and methods for their construction.

(2) A systematic procedure for realizing a prescribed impulse response matrix as a system of differential equations in state-variable form.

(3) Construction of unilateral networks from two-terminal, time-variable resistors, inductors and capacitors.

1.3 Summary

In Chapter 2 some pertinent information from the theory of linear systems is reviewed and the various degrees of controllability and observability are defined. The known criteria for controllability and observability are also summarized in this chapter.

New criteria for controllability and observability are derived in Chapter 3. These criteria are based on the rank of the "controllability" and "observability" matrices, which are formed from the system coefficient matrices and a finite number of their derivatives. The precise extent to which these matrices characterize controllability and observability is made clear by relating them to a generalized Wronskian matrix of vector functions. It is shown that this matrix has all the important properties possessed by the common Wronskian of scalar functions [24] .

Reducibility is also treated in Chapter 3. The controllability and observability matrices are used not only to test for reducibility but in the actual construction of reduced equivalent systems. The methods presented for system reduction are shown to have several important advantages in comparison with previous techniques that were restricted to either fixed systems [18] or a special class of time-variable systems [25]. The basis of the present reduction scheme is an interesting transformational property of the controllability and observability matrices. This property is quite evident once observed, but does not seem to have been exploited previously, even for fixed systems.

Further application of this property is made in Chapter 4, where a general approach to the problems of system equivalence and representation is presented. A new degree of controllability and observability is defined in this chapter. For the class of systems possessing these properties (which includes fixed controllable and observable systems), necessary and sufficient conditions for system equivalence are derived, together with explicit methods for constructing transformations relating equivalent systems. Among the interesting and potentially useful results obtained from this investigation is a criterion for determining if a system is zero-state equivalent to a fixed system. Also given in Chapter 4 are methods for transforming single-input-single-output time-variable systems

to several important canonical equivalents. Included are the classical input-output differential equation form, and the "phase-variable" canonical form [26-28], widely used in control system applications. These canonical structures are especially valuable when simulating a time-variable system on an analogue (or digital) computer since they require relatively few variable components.

The theory of equivalent systems developed in Chapter 4 provides the basis for a new approach to the problem of synthesizing a prescribed impulse response matrix as a system of differential equations (i.e., an analogue computer realization). This method, presented in Chapter 5, differs significantly from previous techniques [18, 23, 29, 30] in that an a priori assumption of realizability is not required. While this method is not completely general, it provides a systematic synthesis procedure for a wide class of responses. In particular, for stationary impulse response matrices, a theory and procedure for minimal realization is obtained which has many advantages in comparison with previous approaches [18, 30, 31, 32].

In Chapter 6, application of the controllability criteria is made to a problem concerning time-variable networks. While it is well known that networks containing time-variable RLC components are generally non-reciprocal, the possibility that such networks may in fact exhibit unilateral behavior does not appear to have been explored.

By relating the system concept of controllability to the network concept of unilateral transmission, we are able to systematically generate a class of unilateral RC networks.

References

[1] L. A. Zadeh, "Time-Varying Networks, 1, " Proc. IRE, vol. 49, No. 10, pp. 1488-1502; Oct. 1961.

[2] W. R. Bennett, "A General Review of Linear Varying Parameter and Non-linear Circuit Analysis." Proc. IRE, vol. 38, pp. 259-263; March 1950.

[3] L. A. Zadeh and C. A. Desoer, Linear System Theory, McGraw-Hill Book Co., Inc., New York, N. Y.; 1963.

[4] M. Athans, "The Status of Optimal Control Theory and Applications for Deterministic Systems," IEEE Conv. Record, vol. 14, part 6, pp. 100-124; 1966.

[5] T. R. Bashkow, "The A Matrix, a New Network Description," IRE Trans. on Circuit Theory, vol. CT-4, pp. 117-119; Sept., 1957.

[6] B. K. Kinariwala, "Analysis of Time-Varying Networks," IRE Conv. Record, vol. 9, part 4, 1961, pp. 268-276.

[7] P. R. Bryant, "The Explicit Form of Bashkow's A-Matrix, IRE Trans. on Circuit Theory, vol. CT-9, pp. 303-306; Sept., 1962.

[8] H. E. Meadows, "Analysis of Linear Time-Variable RC Networks," IEEE Conv. Record, vol. 11, part 2, pp. 34-42; 1963.

[9] C. A. Desoer and A. Paige, "Linear Time Varying G-C Networks: Stable and Unstable," IEEE Trans. on Circuit Theory, vol. CT-10, pp. 180-190; June, 1963.

[10] E. S. Kuh, "Stability of Linear Time-Varying Networks-The State Space Approach," IEEE Trans. on Circuit Theory, vol. CT-12, pp. 150-156; June, 1965.

[11] E. S. Kuh and R. A. Rohrer, "The State-Variable Approach to Network Analysis," Proc. IEEE, vol. 53, pp. 672-686; July, 1965.

[12] L. S. Pontryagin, "Optimal Control Processes," Uspekhi Matem. Nauk., vol. 14, pp. 3-20; 1959 (English translation in American Math. Soc. Trans., Ser. 2, vol. 18, pp. 321-339; 1961.)

[13] R. V. Gamkrelidze, "Theory of Time-Optimal Processes for Linear System," Isv. Akad. Nauk., USSR, Ser. Mat., vol. 2 pp. 449-474.

[14] J. P. LaSalle, "The Time-Optimal Control Problem," Contributions to Nonlinear Oscillations, Princeton University Press, Princeton, N. J., vol. 5, pp. 1-24; 1960.

[15] J. E. Bertram and P. E. Sarachik, "On Optimal Computer Control," Proc. 1st International Congress on Automatic Control, Buttersworths Scientific Publications, London, 1961, vol. 1, pp. 979-982.

[16] R. E. Kalman, "On the General Theory of Control Systems," Proc. 1st Internat'l. Congress on Automatic Control, Buttersworths Scientific Publications; London, England, vol. 1 pp. 481-492; 1961.

[17] R. E. Kalman, "Canonical Structure of Linear Dynamical Systems," Proc. Nat'l. Acad. Sci., U. S., vol. 48, pp. 596-600; April, 1962.

[18] R. E. Kalman, "Mathematical Description of Linear Dynamical Systems," J. SIAM Control, vol. 1, pp. 152-192; 1963.

[19] R. E. Kalman, "A New Approach to Linear Filtering and Prediction Problems," ASME Trans. (J. of Basic Eng.), vol. 82D, pp. 35-45; March, 1960.

[20] R. E. Kalman and R. S. Bucy, "New Results in Linear Filtering and Prediction Theory," ASME Trans. (J. of Basic Eng.), vol. 83D, pp. 95-108; March, 1961.

[21] E. Kreindler and P. E. Sarachik, "On the Concepts of Controllability and Observability of Linear Systems," IEEE Trans. on Automatic Control, vol. AC-9, pp. 129-136; April, 1964.

[22] R. E. Kalman, Y. C. Ho and K. S. Narendra, "Controllability of Linear Dynamical Systems," Contrib. Differential Equations, vol. 1, pp. 189-213; 1962.

[23] L. Weiss and R. E. Kalman, "Contributions to Linear System Theory," Technical Report 64-9, Research Institute for Advanced Study, Maryland; March, 1964.

[24] W. Hurewicz, Lectures on Ordinary Differential Equations, M. I. T. Press, Cambridge, Mass.; 1958.

[25] A. R. Stubberud, "Reducibility of Linear Systems," IEEE Trans. on Appl. and Ind., vol. 67, pp. 214-219; July, 1963.

[26] B. S. Morgan, "The Synthesis of Single Variable Systems by State Variable Feedback," Proc. Allerton Conf. on Circuit and System Theory, University of Illinois, Urbana; November, 1963.

[27] C. D. Johnson and J. E. Gibson, "Optimal Control with Quadratic Performance Index and Fixed Terminal Time," IEEE Trans. on Automatic Control, vol. AC-9, pp. 355-360; October, 1964.

[28] R. W. Bass and I. Gura, "High Order System Design Via State-Space Considerations," Preprints JACC, Troy, N. Y.; June, 1965.

[29] D. C. Youla, "The Synthesis of Linear Dynamical Systems from Prescribed Weighting Patterns," PIB Report PIB MRI-1271-65, Department of Electrophysics Polytechnic Institute of Brooklyn, June 1965.

[30] C. A. Desoer, "The Minimal Realization of Impulse Response Matrices," ERL Tech Memorandum M-128, University of California, Berkeley; September, 1965.

[31] E. G. Gilbert, "Controllability and Observability in Multi-Variable Control Systems," J. SIAM Control, vol. 1, pp. 128-151; 1963.

[32] R. E. Kalman, "Irreducible Realizations and the Degree of a Rational Matrix," J. SIAM, vol. 13, pp. 520-545; June, 1965.

CHAPTER 2

TIME-VARIABLE LINEAR SYSTEMS

2.1 Basic Definitions and Notation

The class of systems to be considered in this thesis are those describable by a finite set of first order differential equations of the form*

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{2.1a}$$

$$y(t) = C(t)x(t) , \tag{2.1b}$$

where $x(t)$, an n-vector, is the state of the system at time t ; $u(t)$, an r-vector, is the input; and $y(t)$, an m-vector, is the output. The matrices $A(t)$, $B(t)$ and $C(t)$, are of order compatible with the vectors $x(t)$, $u(t)$ and $y(t)$.

*Notation: Lower case symbols will be used to denote both scalar and vector functions, while upper case symbols will be reserved for all other matrices. When the context is clear, the explicit dependence of functions on their argument will be suppressed (i.e., $A = A(t)$). The operations of transposition and inversion will be denoted by A' and A^{-1}, respectively, and the following notations will be used for differentiation:

$$\frac{d}{dt}A = \dot{A} = A^{(1)} ,$$

$$\frac{d^k}{dt^k}A = A^{(k)} .$$

If the coefficient matrices are sufficiently well behaved,*

the output of such a system is given by [1]

$$y(t) = C(t)\Phi(t, t_0)x(t_0) + \int_{t_0}^{t} C(t)\Phi(t, \tau)B(\tau)u(\tau)d\tau \ , \qquad (2.2)$$

where $x(t_0)$ is the state of the system at some arbitrary time t_0, and $\Phi(t, \tau)$ is the system transition matrix. The transition matrix may be defined as

$$\Phi(t, \tau) = X(t)X^{-1}(\tau) \qquad (2.3)$$

where $X(t)$, an $n \times n$ matrix, is a fundamental matrix [2] of (2.1a); that is,

$$\dot{X}(t) = A(t)X(t) \qquad (2.4)$$

and $X(t)$ has rank n for all t.

It is also convenient to define the impulse response matrix of system (2.1):

---

*To avoid unnecessary complication, it will generally be assumed that the matrices A, B and C are continuous functions of time.

$$H(t,\tau) = \begin{cases} C(t)\Phi(t,\tau)B(\tau), & t \geq \tau \\ 0 \quad , & t < 0 \end{cases} \tag{2.5}$$

For zero initial conditions, the impulse response matrix completely characterizes the input-output behavior of system (2.1) since if $x(t_0) = 0$,

$$y(t) = \int_{t_0}^{t} H(t,\tau)u(\tau)d\tau \; . \tag{2.6}$$

From (2.3) it is clear that the impulse response matrix may also be expressed in the form

$$H(t,\tau) = \Psi(t)\Theta(\tau) \, , \qquad t \geq \tau \tag{2.7}$$

where

$$\Psi(t) = C(t)X(t) \; , \tag{2.8}$$

$$\Theta(\tau) = X^{-1}(\tau)B(\tau) \; , \tag{2.9}$$

and $X(t)$ is a fundamental matrix of (2.1a). Although $\Psi(t)$ and $\Theta(t)$ are not unique matrices for a given system, they are unique within a constant transformation. If $X_1(t)$ and $X_2(t)$ are any two funda-

mental matrices of (2.1a) then

$$X_1(t) = X_2(t)K \tag{2.10}$$

where K is a constant nonsingular matrix [2] .

It is often useful to consider various equivalent representations of a given system for purposes of both analysis and synthesis. There are many types of system equivalence that can be defined [3, 4], but the following will be most important here.

Definition 2.1: Let $T(t)$ be an $n \times n$ matrix, nonsingular and continuously differentiable for all t, and let $z(t) = T(t)x(t)$. Then it will be said that the system

$$\dot{z}(t) = \bar{A}(t)z(t) + \bar{B}(t)u(t) \tag{2.11a}$$

$$y(t) = \bar{C}(t)z(t) \tag{2.11b}$$

where

$$\bar{A} = (TA + \dot{T})T^{-1} \tag{2.12a}$$

$$\bar{B} = TB \tag{2.12b}$$

$$\bar{C} = CT^{-1} , \tag{2.12c}$$

is equivalent (algebraically equivalent [3]) to system (2.1) and that T is an equivalence transformation.

If $X(t)$ is a fundamental matrix of (2.1a) then $T(t)X(t)$ is clearly a fundamental matrix of (2.11a), so that the transition matrix of (2.11) is given by

$$\overline{\Phi}(t,\tau) = T(t)\Phi(t,\tau)T^{-1}(\tau) \quad . \tag{2.13}$$

The impulse response matrix is thus seen to be invariant under an equivalence transformation since from (2.12) and (2.13)

$$\begin{aligned}\overline{H}(t,\tau) &= \overline{C}(t)\overline{\Phi}(t,\tau)\overline{B}(\tau) \\ &= C(t)\Phi(t,\tau)B(\tau) = H(t,\tau) \ .\end{aligned}$$

Furthermore, if $x(t_0)$ is the initial state of (2.1), and $z(t_0) = T(t_0)x(t_0)$ is the initial state of (2.11) then

$$\overline{C}(t)\overline{\Phi}(t,t_0)z(t_0) = C(t)\Phi(t,t_0)x(t_0) \ .$$

Thus, systems equivalent under Definition 2.1 are both zero-state and zero-input equivalent [4] .

2.2 Controllability

The concept of controllability as it arose in the study of optimal control problems is essentially concerned with the possible state transitions that can be effected in a system by application of some input. There are two basic types or "degrees" of controllability having importance in control theory; complete [3, 5, 6], and total [7] (differential [8], proper [9]) controllability. The major difference between the two is that total controllability insures that the state of the system can be controlled as quickly as desired, while complete controllability only implies that the state can be controlled in some finite time.

Precise definitions of these concepts will be given below. Various equivalent definitions, as well as a fuller discussion of the role they play in control theory may be found in the work of Kalman [3, 5], LaSalle [9], Kalman, Ho and Narendra [6], Kreindler and Sarachik [7], and Weiss and Kalman [8].

Definition 2.2 (Controllability):

(a) System (2.1) is said to be completely controllable on an interval $[t_0, t_1]$ if for any state x_0 at t_0, and any desired final state x_1 at t_1, there exists an input $u(t)$ defined on $[t_0, t_1]$ such that $x(t_1) = x_1$.

(b) System (2.1) is said to be completely controllable at time t_0 if there exists a finite time $t_1 > t_0$ such that the system is completely

controllable on $[t_0, t_1]$. If system (2.1) is completely controllable at all t_0, it is said to be completely controllable.

(c) System (2.1) is said to be totally controllable on an interval $[t_0, t_1]$ if it is completely controllable on every subinterval of $[t_0, t_1]$.

(d) System (2.1) is said to be totally controllable at time t_0 if for all $t_1 > t_0$, it is completely controllable on $[t_0, t_1]$. If system (2.1) is totally controllable at all t_0, it is said to be totally controllable.

Necessary and sufficient conditions for controllability of a time-variable system are well known [6, 7, 8] and will be summarized below. If $X(t)$ is any fundamental matrix for (2.1a) and $\Theta(t) = X^{-1}(t)B(t)$, then

Theorem 2.1: System (2.1) is completely controllable on the interval $[t_0, t_1]$ if and only if the rows of $\Theta(\tau)$ are linearly independent functions of τ on $[t_0, t_1]$.

Theorem 2.2: System (2.1) is totally controllable on the interval $[t_0, t_1]$ if and only if the rows of $\Theta(\tau)$ are linearly independent functions of τ on every subinterval of $[t_0, t_1]$.

Criteria for controllability at a specific initial time, and for all initial times follow naturally from Definition 2.1 and the above theorems [7, 8].

It should be observed that it is usual to express the controllability criteria in terms of particular fundamental matrices

(either $\Phi(t_1, \tau)$ [7] or $\Phi(t_0, \tau)$ [8]). Since any two fundamental matrices are related by a constant (with respect to τ), nonsingular matrix as in (2.10), the independence of the rows of $\Theta(\tau)$ is not affected by the choice of fundamental matrix used to define it.

Furthermore, controllability is invariant under an equivalence transformation. To see this, let $\overline{X}(t)$ be a fundamental matrix for (2.11a) and let

$$\overline{\Theta}(t) = \overline{X}^{-1}(t)\overline{B}(t) .$$

By definition, $\overline{B}(t) = T(t)B(t)$, and as noted previously, $\overline{X}(t) = T(t)X(t)$, where $X(t)$ is a fundamental matrix for (2.1a). Therefore, $\overline{\Theta}(t) = \Theta(t)$ and anything said about the controllability of (2.1), must hold for (2.11).

2.3 Observability:

The concept of observability introduced by Kalman [3, 5, 10], is of considerable importance in optimal filtering and prediction problems. Essentially, if a system is observable, its state at some particular time can be determined from observations of the system output. Various degrees of observability will be defined below. It will be clear from these definitions and the criteria which follow that observability plays a role dual to that of controllability in describing a systems structure. This duality first observed by Kalman [10]

greatly simplifies the solution of many problems [5, 10]. Further discussion of observability and the dual relationship it bears to controllability may be found in [7] and [8].

Definition 2.3 (Observability):

(a) System (2.1) is said to be completely observable on an interval $[t_0, t_1]$ if any initial state x_0 at t_0 can be determined from knowledge of the system's output over $[t_0, t_1]$.

(b) System (2.1) is said to be completely observable at time t_0 if there exists a finite time $t_1 > t_0$ such that the system is completely observable on $[t_0, t_1]$. If system (2.1) is completely observable at all t_0, it is said to be completely observable.

(c) System (2.1) is said to be totally observable on an interval $[t_0, t_1]$ if it is completely observable on every subinterval of $[t_0, t_1]$.

(d) System (2.1) is said to be totally observable at time t_0 if for all $t_1 > t_0$, it is completely observable on $[t_0, t_1]$. If system (2.1) is totally observable at all t_0, it is said to be totally observable.

If $X(t)$ is any fundamental matrix for (2.1a) and $\Psi(t) = C(t)X(t)$ then the following conditions for observability may be established [7, 8].

Theorem 2.3: System (2.1) is completely observable on the interval $[t_0, t_1]$ if and only if the columns of $\Psi(t)$ are linearly independent functions of t on $[t_0, t_1]$.

<u>Theorem 2.4</u>: System (2.1) is totally observable on the interval $[t_0, t_1]$ if and only if the columns of $\Psi(t)$ are linearly independent functions of t on ever subinterval of $[t_0, t_1]$.

The form of Theorems 2.3 and 2.4 when compared with that of Theorems 2.1 and 2.2 immediately suggests a precise formulation of the duality between observability and controllability [7].

<u>Theorem 2.5</u>: System (2.1) is completely controllable (observable) on the interval $[t_0, t_1]$ if and only if the system

$$\dot{w}(t) = -A'(t)w(t) + C'(t)\tilde{u}(t) \tag{2.14a}$$

$$\tilde{y}(t) = B'(t)w(t) \tag{2.14b}$$

is completely observable (controllable) on $[t_0, t_1]$.

To prove the theorem, it is only necessary to observe that if $X(t)$ is a fundamental matrix of (2.1a), then $(X^{-1}(t))'$ is a fundamental matrix of (2.14a) [2].

If system (2.1) is time-invariant, all degrees of controllability (and observability) are equivalent and we may classify systems as being either controllable (observable) or non controllable (non observable). Most importantly, the necessary and sufficient conditions for controllability and observability in the fixed case can be formulated directly in terms of the coefficient matrices of the system, as in the following theorems [3].

Theorem 2.6: A fixed system of the form (2.1) is completely controllable if and only if the matrix Q_c has rank n, where

$$Q_c = [B \vdots AB \vdots \ldots \vdots A^{n-1}B] \; . \tag{2.15}$$

Theorem 2.7: A fixed system of the form (2.1) is completely observable if and only if the matrix Q_0 has rank n, where

$$Q_0 = [C' \vdots A'C' \vdots \ldots \vdots (A^{n-1})'C'] \; . \tag{2.16}$$

In Chapter 3, criteria which seem to be natural generalizations of Theorems 2.6 and 2.7 for time-variable systems will be derived.

References

[1] R. J. Schwarz and B. Friedland, Linear Systems, McGraw-Hill Book Co., Inc., New York, N. Y.; 1965.

[2] E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill Book Co., Inc., New York, N. Y.; 1955.

[3] L. A. Zadeh and C. A. Desoer, Linear System Theory, McGraw-Hill Book Co., Inc., New York, N. Y.; 1963.

[4] R. E. Kalman, "Mathematical Description of Linear Dynamical Systems," J. SIAM Control, vol. 1, pp. 152-192; 1963.

[5] R. E. Kalman, "On the General Theory of Control Systems," Proc. 1st Internat'l. Congress on Automatic Control, Buttersworths Scientific Publications; London, England, vol. 1, pp. 481-492; 1961.

[6] R. E. Kalman, Y. C. Ho and K. S. Narendra, "Controllability of Linear Dynamical Systems," Contrib. Differential Equations, vol. 1, pp. 189-213; 1962.

[7] E. Kreindler and P. E. Sarachik, "On the Concepts of Controllability and Observability of Linear Systems," IEEE Trans. on Automatic Control, vol. AC-9, pp. 129-136; April, 1964.

[8] L. Weiss and R. E. Kalman, "Contributions to Linear System Theory," Technical Report 64-9, Research Institute for Advanced Study, Maryland; March, 1964.

[9] J. P. LaSalle, "The Time-Optimal Control Problem," Contributions to Nonlinear Oscillations, Princeton University Press, Princeton, N. J., vol. 5, pp. 1-24; 1960.

[10] R. E. Kalman, "A New Approach to Linear Filtering and Prediction Problems," ASME Trans. (J. of Basic Eng.), vol. 82D, pp. 35-45; March, 1960.

CHAPTER 3

CONTROLLABILITY, OBSERVABILITY AND REDUCIBILITY*

3.1 Introduction

The necessary and sufficient conditions for controllability and observability summarized in Chapter 2 depend explicitly on the system solution matrix in the time-variable case. Since this matrix is generally not available except as a numerically tabulated solution, it has been difficult to characterize the structural properties of a time-variable system in terms of its coefficient matrix description.

In order to circumvent the problem of system solution we propose to determine the extent to which the controllability and observability properties can be characterized in terms of the matrices A, B and C. This problem has been studied by several other workers including Stubberud [1], and Chang [2]. Our independently derived results will be described below and compared with those of the above.

Of prime importance in this development are the system "controllability" and "observability" matrices. It will be shown that these matrices, which do not require knowledge of the system solution for their construction, provide significant structural information,

---

*Parts of this chapter have appeared in two papers by the author and H. E. Meadows [3, 4].

including: (i) a sufficient condition for complete controllability and observability, and (ii) a necessary and sufficient condition for total controllability and observability. These conditions are established here by relating the controllability and observability matrices to a new test for linear independence of certain vector functions. This test is a generalization of the familiar Wronskian determinant criterion for scalar functions.

The controllability and observability matrices will also be shown to be useful in determining whether a system is non-controllable or non-observable and in constructing transformations of coordinates to reduce such systems to zero-state equivalents of lower dynamic order.

3.2 The Controllability and Observability Matrices

The controllability and observability matrices of a time-variable linear system of the form (2.1) will now be defined. To insure that these matrices exist and are well behaved for all time it will be assumed that the matrices A, B and C are continuously differentiable n-1, n and n times, respectively. This restriction can be considerably weakened for many arguments [3], but at the expense of greater mathematical complication.

The controllability matrix of system (2.1) is defined as

$$Q_c = [P_0 \vdots P_1 \vdots \cdots \vdots P_{n-1}] \tag{3.1}$$

where

$$P_{k+1} = -AP_k + \frac{d}{dt}P_k, \qquad P_0 = B. \tag{3.2}$$

The <u>observability matrix</u> is defined in a dual manner as

$$Q_o = [S_0 \vdots S_1 \vdots \cdots \vdots S_{n-1}] \tag{3.3}$$

$$S_{k+1} = A'S_k + \frac{d}{dt}S_k, \qquad S_0 = C'. \tag{3.4}$$

Whenever the context is clear, the subscripts 'c' and 'o' will be dropped.

A property of these matrices which proves to be quite useful in the subsequent development concerns their behavior under transformation of system coordinates.

<u>Property 3.1</u>: If T is an equivalence transformation and if $\hat{Q}_c$ is the controllability matrix of the transformed system, then

$$\hat{Q}_c = TQ_c. \tag{3.5}$$

Equation (3.5) follows by induction since if $\hat{P}_k = TP_k$ then

$$\begin{aligned} \hat{P}_{k+1} &= -\hat{A}\hat{P}_k + \frac{d}{dt}\hat{P}_k \\ &= (TA + \dot{T})T^{-1}(TP_k) + (\dot{T}P_k + T\dot{P}_k) \\ &= T(-AP_k + \dot{P}_k) = TP_{k+1} \end{aligned}$$

and by definition,

$$\hat{P}_0 = \hat{B} = TB = TP_0 .$$

Similarly, if $\hat{Q}_0$ is the observability matrix of the transformed system then

$$\hat{Q}_0 = (T')^{-1} Q_0 . \tag{3.6}$$

To indicate the role played by the controllability matrix as a test for controllability it is first noted that

$$\frac{d^k}{d\tau^k}[X^{-1}(\tau)B(\tau)] = X^{-1}(\tau)P_k(\tau) \tag{3.7}$$

where $X(\tau)$ is any fundamental matrix of (2.1a). Equation (3.7) follows by induction since $P_0 = B$, and if

$$\frac{d^k}{d\tau^k}[X^{-1}(\tau)B(\tau)] = X^{-1}(\tau)P_k(\tau) ,$$

then

$$\frac{d^{k+1}}{d\tau^{k+1}}[X^{-1}(\tau)B(\tau)] = \frac{d}{d\tau}[X^{-1}(\tau)P_k(\tau)]$$

$$= -X^{-1}(\tau)\dot{X}(\tau)X^{-1}(\tau)P_k(\tau)+X^{-1}(\tau)\dot{P}_k(\tau).$$

but

$$\dot{X}(\tau)X^{-1}(\tau) = A(\tau) .$$

Therefore,

$$\frac{d^{k+1}}{d\tau^{k+1}}[X^{-1}(\tau)B(\tau)] = X^{-1}(\tau)[-A(\tau)P_k(\tau) + \dot{P}_k(\tau)]$$

$$= X^{-1}(\tau)P_{k+1}(\tau) .$$

It follows then that

$$[\Theta(\tau) \vdots \Theta^{(1)}(\tau) \vdots \ldots \vdots \Theta^{(n-1)}(\tau)] = X^{-1}(\tau)Q_c(\tau) \qquad (3.8)$$

where $\Theta(t) = X^{-1}(t)B(t)$. If u is a scalar input then the matrix on the left side of (3.8) is recognized to be the Wronskian matrix of the rows of $\Theta(\tau)$. It is well known [5] that the rank of the Wronskian matrix may be used to test the linear independence of scalar functions. Since $X^{-1}(\tau)$ is nonsingular for all τ, the rank of $Q_c(\tau)$ is equal to that of the Wronskian matrix for all τ. For single input systems, therefore, the controllability matrix yields as much information about the degree of controllability of the system as does the Wronskian matrix of the rows of $\Theta(\tau)$. In the following section, it will be shown that a Wronskian matrix may be defined for vector functions having all the important properties of the scalar Wronskian matrix.

3.3 The Vector Wronskian Matrix

Consider the set of r-dimensional row vector functions $\theta_1(t), \theta_2(t), \ldots, \theta_n(t)$, where the elements of each $\theta_i(t)$ together with their first n derivatives are continuous functions. The Wronskian matrix of such a set of functions is defined as

$$W(\theta_1, \theta_2, \ldots, \theta_n) = [\Theta_0 \vdots \Theta_1 \vdots \cdots \vdots \Theta_{n-1}] \qquad (3.9)$$

where

$$\Theta_0 = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \cdot \\ \cdot \\ \cdot \\ \theta_n \end{bmatrix} = \Theta, \qquad \Theta_k = \frac{d}{dt}\Theta_{k-1}.$$

When the context is clear $W(t)$ will denote the Wronskian matrix.

The following theorem provides a direct generalization of the test for linear independence of scalar functions.

Theorem 3.1: If $W(t)$ has rank n for some t in an interval $[t_0, t_1]$, then the functions $\theta_1, \theta_2, \ldots, \theta_n$ are linearly independent on $[t_0, t_1]$.

To prove the theorem, suppose that the θ_i are dependent on $[t_0, t_1]$; then there exists a constant n-vector $\lambda \neq 0$ such that for all $t \epsilon [t_0, t_1]$,

$$\lambda' \Theta(t) = 0$$

By differentiating this relationship n-1 times, it is clear that for all $t \in [t_0, t_1]$,

$$\lambda' W(t) = 0$$

which contradicts $W(t)$ having rank n for some $t \in [t_0, t_1]$.

The following theorem provides an important test for linear dependence of vector functions. Although the statement of the theorem is a direct generalization of a known result for scalar functions [5], the proof is believed to be entirely novel in that it utilizes properties of forced linear systems of the form (2.1). The usual proof of this theorem in the scalar case [5] relies on the relationship the Wronskian of a set of functions bears to the solution of homogeneous differential equations - no such analogous relationship holds for vector functions.

Theorem 3.2: If $W(t)$ has rank q less than n for all $t \in [t_0, t_1]$ and if the Wronskian matrix of any q of the θ_i has rank q for all $t \in [t_0, t_1]$, then the functions $\theta_1, \theta_2, \ldots, \theta_n$ are linearly dependent on $[t_0, t_1]$, and may be expressed as a linear combination of the q independent rows.

Consider the system

$$\begin{aligned} \dot{x} &= \Theta(t)u \\ y &= x \end{aligned} \tag{3.10}$$

Let Q be the controllability matrix of (3.10); then it is clear that $Q = W$. Without loss of generality, it may be assumed that $\theta_1, \theta_2, \ldots, \theta_q$ have a Wronskian matrix of rank q for all $t \epsilon [t_0, t_1]$, then

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix},$$

where Q_1 has q rows and rank q for all $t \epsilon [t_0, t_1]$. Since Q also has rank q everywhere on the same interval, the rows of Q_2 can be written as a linear combination of the rows of Q_1 for all $t \epsilon [t_0, t_1]$; that is

$$Q_2 = KQ_1 \tag{3.11}$$

where K is an $(n-q) \times q$ matrix. Note that

$$K = Q_2 Q_1^{\dagger}$$

where $Q_1^{\dagger} = Q_1'(Q_1 Q_1')^{-1}$ so that K has a continuous derivative.

If it can be shown that K is a constant matrix the theorem will be established.

Define the matrix

$$T = \begin{bmatrix} I_q & 0 \\ -K & I_{n-q} \end{bmatrix}, \tag{3.12}$$

where I_q is the q^{th} order identity matrix, and note that

$$T\Theta_k = \begin{bmatrix} \Omega_k \\ \\ 0 \end{bmatrix}, \qquad k = 0, 1, \ldots, n-1$$

where

$$\Omega_k = \frac{d}{dt}\Omega_{k-1}, \qquad \Omega_0 = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_q \end{bmatrix} = \Omega .$$

T is clearly nonsingular for all t and as K is continuously differentiable on $[t_0, t_1]$, T is an equivalence transformation on the interval. If $z = Tx$, therefore, the system

$$\begin{aligned} \dot{z} &= \hat{A}z + \hat{B}u \\ y &= \hat{C}z \end{aligned} \qquad (3.13)$$

where

$$\begin{aligned} \hat{A} &= \dot{T}T^{-1} \\ \hat{B} &= T\Theta \\ \hat{C} &= T^{-1} \end{aligned}$$

is equivalent to (3.10) on $[t_0, t_1]$. Since

$$T^{-1} = \begin{bmatrix} I_q & 0 \\ \\ K & I_{n-q} \end{bmatrix},$$

then

$$\hat{A} = \begin{bmatrix} 0 & 0 \\ -\dot{K} & 0 \end{bmatrix} .$$

It is also true, however, from the definition of P_i that

$$\hat{A}[\hat{P}_0 \vdots \cdots \vdots \hat{P}_{q-1}] = \frac{d}{dt}[\hat{P}_0 \vdots \cdots \vdots \hat{P}_{q-1}] - [\hat{P}_1 \vdots \cdots \vdots \hat{P}_q]$$

or

$$\begin{bmatrix} 0 & 0 \\ -\dot{K} & 0 \end{bmatrix} \begin{bmatrix} \Omega_0 \vdots \cdots \vdots \Omega_{q-1} \\ 0 \vdots \cdots \vdots 0 \end{bmatrix} = \begin{bmatrix} \dot{\Omega}_0 - \Omega_1 & \vdots \cdots \vdots & \dot{\Omega}_{q-1} - \Omega_q \\ 0 & \vdots \cdots \vdots & 0 \end{bmatrix}$$

Therefore,

$$\dot{K}[\Omega_0 \vdots \cdots \vdots \Omega_{q-1}] = 0$$

or

$$\dot{K}\, W(\theta_1, \theta_2, \ldots, \theta_q) = 0 \tag{3.14}$$

By assumption, $W(\theta_1, \theta_2, \ldots \theta_q)$ has rank q for all $t \epsilon [t_0, t_1]$ so that (3.14) implies

$$\dot{K} = 0 ,$$

which establishes the theorem.

The following corollary to Theorem 3.2 provides a test for determining whether a set of functions are dependent over some interval.

Corollary 3.1: If the Wronskian matrix has rank less than n for all $t\epsilon[t_0, t_1]$ then there exists some subinterval of $[t_0, t_1]$ over which the functions $\theta_1, \theta_2, \ldots, \theta_n$ are linearly dependent.

The proof of this corollary by induction is a generalization of that for the scalar case given by Hurewicz [5].

For $n = 1$, $W(\theta_1) = \theta_1$ so that if the rank of $W(\theta_1)$ is less than one for all $t\epsilon[t_0, t_1]$ then $\theta_1 = 0$ for all $t\epsilon[t_0, t_1]$ and the corollary is trivially true.

Suppose that the corollary is valid for $n = k-1$ and $W(\theta_1, \theta_2, \ldots, \theta_k)$ has rank less than k for all $t\epsilon[t_0, t_1]$. Either $W(\theta_1, \theta_2, \ldots, \theta_{k-1})$ has rank $k-1$ for some $t\epsilon[t_0, t_1]$ or it has rank less than $k-1$ for all $t\epsilon[t_0, t_1]$. If the former is true, then by continuity there must exist some subinterval of $[t_0, t_1]$ on which $W(\theta_1, \theta_2, \ldots, \theta_{k-1})$ has rank $k-1$ everywhere. In this case, Theorem 3.2 implies that $\theta_1, \theta_2, \ldots, \theta_k$ are dependent on the subinterval. If $W(\theta_1, \theta_2, \ldots \theta_{k-1})$ has rank less than $k-1$ for all $t\epsilon[t_0, t_1]$ then by assumption $\theta_1, \theta_2, \ldots, \theta_{k-1}$ are linearly dependent on some subinterval of $[t_0, t_1]$, which completes the proof.

It is now possible to prove the major result for the vector Wronskian matrix which, when related to the controllability matrix, provides a necessary and sufficient condition for total controllability.

Theorem 3.3: A necessary and sufficient condition for the row vector functions $\theta_1, \theta_2, \ldots, \theta_n$ to be linearly independent on every subinterval of an interval $[t_0, t_1]$ is that their Wronskian matrix not have rank less than n on any subinterval of $[t_0, t_1]$.

Sufficiency follows immediately from Theorem 3.1 since any subinterval of $[t_0, t_1]$ must contain points where $W(t)$ has rank n.

Necessity follows from Corollary 3.1 since if $W(t)$ has rank less than n on some subinterval of $[t_0, t_1]$, there must be some sub-subinterval over which the functions are dependent.

If the elements of the row vectors $\theta_1, \theta_2, \ldots, \theta_n$ are analytic functions of t the Wronskian test for linear independence can be strengthened considerably. In the Appendix properties of matrices of analytic functions are discussed, and the following result which follows easily from Theorem 3.2 is established.

Corollary 3.2: Let $\theta_1, \theta_2, \ldots, \theta_n$ be row vectors of functions analytic on $[t_0, t_1]$; then $\theta_1, \theta_2, \ldots, \theta_n$ are linearly independent on $[t_0, t_1]$ if and only if $W(t)$ has rank n for some $t \in [t_0, t_1]$. Furthermore, if $W(t)$ has maximal rank q less than n, exactly q of the θ_i are linearly independent on the interval.

It should be noted that the rank of a matrix of analytic functions is constant save possibly at a finite number of points in any finite interval. Thus if $W(t)$ has rank n for some $t \in [t_0, t_1]$ it has rank n for all but a finite number of points in

$[t_0, t_1]$.* It is clear then that if a set of analytic row vectors are independent on any subinterval of $[t_0, t_1]$ they are independent on every subinterval of $[t_0, t_1]$.

3.4 Criteria for Controllability and Observability

If $\Theta(t) = X^{-1}(t)B(t)$ where $X(t)$ is any fundamental matrix solution of (2.1) and $W_c(t)$ is the Wronskian matrix of the rows of $\Theta(t)$ then from (3.8)

$$W_c(t) = X^{-1}(t)Q_c(t) \tag{3.15}$$

Immediate application of Theorem 3.1 gives the following sufficient condition for complete controllability.

Theorem 3.4: (a) System (2.1) is completely controllable on the interval $[t_0, t_1]$ if Q_c has rank n for some $t \in [t_0, t_1]$.

(b) If for some $t > t_0$, Q_c has rank n then system (2.1) is completely controllable at t_0.

---

*To simplify notation, if an analytic matrix has rank q at all but a finite number of points on an interval, we will say that the matrix "has rank q" on the interval.

This result was also prov. directly in [3] and independently by Stubberud [1], and Chang [2], and is the time-variable form of Pontryagin's "general position criteria" [6]. It was also shown in [3] that Theorem 3.4 holds under much weaker conditions on the coefficient matrices.

In general the condition of Theorem 3.4 is not necessary for complete controllability as demonstrated by the following example.

Example 3.1: Consider the second order single-input system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ f(t) & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} g(t) \\ 0 \end{bmatrix} u$$

where

$$f(t) = \begin{cases} 0 & t \le 0 \\ \gamma(t-1) & 0 \le t \le 2 \\ 0 & 2 \le t \le 4 \\ \gamma(t-5) & 4 \le t \le 6 \\ 0 & 6 \le t \end{cases},$$

$$g(t) = f(t-2),$$

and

$$\gamma(t) = \begin{cases} \exp\left(\dfrac{-1}{1-t^2}\right) & |t| \le 1 \\ 0 & |t| \ge 1 . \end{cases}$$

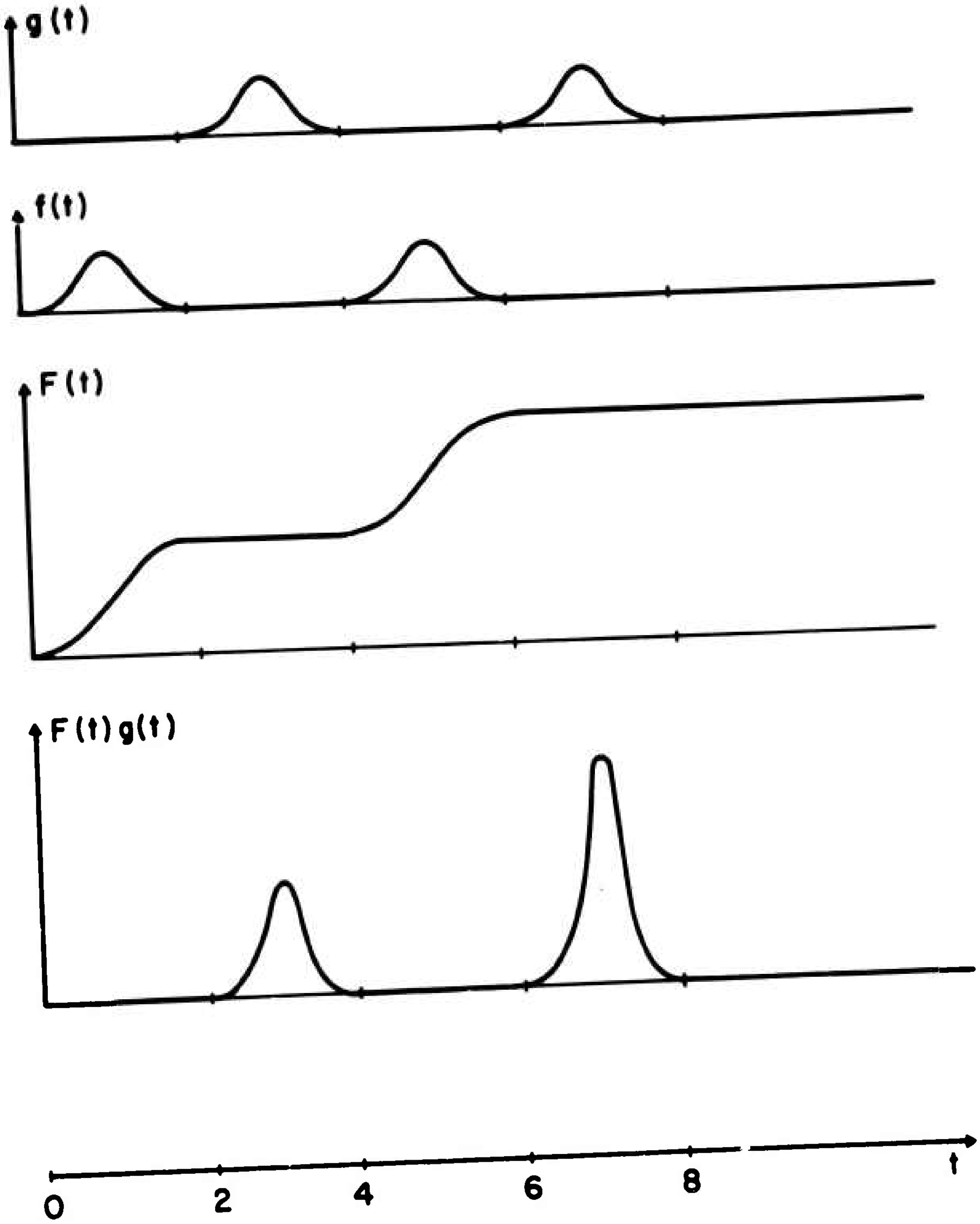


Fig. 3.1 Form of parameters in Example 3.1

A fundamental matrix for this system is

$$X(t) = \begin{bmatrix} 1 & 0 \\ F(t) & 1 \end{bmatrix},$$

where $F(t) = \int_{-\infty}^{t} f(s)ds$, so that

$$\Theta(t) = X^{-1}(t)\,B(t) = \begin{bmatrix} 1 & 0 \\ F(t) & 1 \end{bmatrix}\begin{bmatrix} g(t) \\ 0 \end{bmatrix} = \begin{bmatrix} g(t) \\ F(t)\,g(t) \end{bmatrix}$$

It is clear here from Fig. (3.1) that $g(t)$ and $F(t)g(t)$ are linearly independent on the interval $[0, 8]$. However,

$$Q_c = \begin{bmatrix} g(t) & \dot{g}(t) \\ 0 & -f(t)\,g(t) \end{bmatrix} = \begin{bmatrix} g(t) & \dot{g}(t) \\ 0 & 0 \end{bmatrix},$$

so that Q_c has rank <2 for all $t \in [0, 8]$.

Observe that even though this system is completely controllable on $[0, 8]$ it is not controllable on any interval $[0, t]$ for $t \le 6$. That is, starting at time $t = 0$ there is a waiting period of 6 units before a desired final state can be reached.

Although a necessary and sufficient condition is not possible for complete controllability in terms of the Q_c matrix such a condition can be given for the more useful property of total controllability. This condition, which follows directly from Theorem 3.3 and equation (3.15), constitutes a major result of this chapter. It may be summarized as follows:

<u>Theorem 3.5:</u> System (2.1) is totally controllable on the interval $[t_0, t_1]$ if and only if Q_c does not have rank less than n on any subinterval of $[t_0, t_1]$.

If (2.1) is a fixed system, then the controllability matrix takes the form

$$Q_c = [B \vdots -AB \vdots \ldots \vdots (-A)^{n-1}B]$$

and is equivalent in rank to the familiar form of the fixed controllability matrix given in equation (2.15). Theorem 3.5 may thus be viewed as a generalization of the well known and widely useful criterion for controllability of fixed systems given in Chapter 2.

It should be quite apparent that results dual to those presented for controllability hold for observability. To formalize these results we first define the Wronskian matrix of a set of column vector functions

$$\Psi(t) = [\psi_1(t) \vdots \psi_2(t) \vdots \ldots \vdots \psi_n(t)]$$

as

$$W(t) = \begin{bmatrix} \Psi_0(t) \\ \Psi_1(t) \\ \cdot \\ \cdot \\ \cdot \\ \Psi_{n-1}(t) \end{bmatrix},$$

where

$$\Psi_k(t) = \frac{d}{dt}\Psi_{k-1}(t) , \qquad \Psi_0(t) = \Psi(t) .$$

If $\Psi(t) = C(t)X(t)$ where $X(t)$ is a fundamental matric solution of (2.1), then the matrix

$$W_0(t) = Q_0'X(t) \tag{3.16}$$

plays a role completely dual to that of $W_c(t)$ in establishing the following theorems.

Theorem 3.6: (a) System (2.1) is completely observable on the interval $[t_0, t_1]$ if Q_0 has rank n for some $t \in [t_0, t_1]$.

(b) If for some $t > t_0$, Q_0 has rank n then the system (2.1) is completely observable at t_0.

Theorem 3.7: System (2.1) is totally observable on the interval $[t_0, t_1]$ if and only if Q_0 does not have rank less than n on any subinterval of $[t_0, t_1]$.

If the coefficient matrices of system (2.1)are analytic functions of time,* the distinction between complete and total controllability disappears, as will now be shown. First note that if (2.1) is an analytic system, its fundamental matrices must also be analytic [7]. Thus, it follows from Corollary 3.2 that if the rows of

*For simplicity, we will refer to (2.1) as an "analytic system" in this case.

$X^{-1}(t)B(t)$ are linearly independent functions on some interval of t, they are independent on every interval of t. That is, if the system is completely controllable on some interval, it is totally controllable on the interval. Since the converse is always true, complete and total controllability are equivalent concepts for analytic systems, and controllability is independent of the particular choice of intervals and initial time (as is true for fixed systems). As a special case of Theorem 3.5, therefore, we have the following corollary.*

Corollary 3.3: An analytic system of the form (2.1) is completely controllable if and only if Q_c has rank n.

Similarly we have:

Corollary 3.4: An analytic system of the form (2.1) is completely observable if and only if Q_0 has rank n.

3.5 System Reduction

In this section we will consider the problem of determining from the Q matrix whether a system is non-controllable (i.e., not completely controllable) on an interval. It is clear from Example 3.1 that this is not always possible. Coupled with this problem is that of "system reduction." It is well known [8] that any non-controllable

*This result was also established by Stubberud [1] and Chang [2].

or non-observable system can be reduced to a zero-state equivalent system of lower dynamic order. When possible such reduction is usually desirable in analysis since the resulting system is more easily solved. Moreover, in synthesis, a first step in realizing a prescribed input-output response may result in a system of higher order than necessary. A least order realization can then be obtained by reducing the initial system. This type of synthesis is utilized [8] for fixed systems. The method of reduction given in [8], however, is based on knowledge of the transition matrix and thus cannot readily be generalized to time-variable systems. Reduction of time-variable systems is considered in [9] in the special case of systems realizable by a single-input-single-output differential equations. Presented below is a general approach to the problem of system reduction based on properties of the controllability and observability matrices. The following will be important in this development.

<u>Property 3.2:</u> If Q_c has rank q for all $t \epsilon [t_0, t_1]$ then any matrix of the form

$$[P_0 \vdots P_1 \vdots \ . \ . \ . \ \vdots P_k] \ , \qquad k > n-1$$

has rank q for all $t \epsilon [t_0, t_1]$.

A proof of this property is given in the appendix of [3].

<u>Property 3.3:</u> Let Q_c be the controllability matrix of (2.1) and define the related matrix

$$Q_c^* = [P_1 \vdots P_2 \vdots \ . \ . \ . \ \vdots P_n]. \tag{3.17}$$

Then on any interval where Q_c has rank n, the system A-matrix can be expressed as

$$A = (\dot{Q}_c - Q_c^*)Q_c^\dagger \tag{3.18}$$

Equation (3.18) follows from the observation that

$$Q_c^* = -AQ_c + \dot{Q}_c \tag{3.19}$$

Since $P_0 = B$, knowledge of consistent matrices Q and P_n suffices to determine (2.1) uniquely on any interval where Q has rank n.

It will be said that a system is <u>reducible</u> if for all time, its impulse response matrix may be realized by a system of lower order. Reduction is always possible if a system is not completely controllable and/or not completely observable. In certain degenerate situations a system may be reducible even if it is completely controllable and observable [10], but such cases will not be considered here. The reduction problem will be separated into two parts:

(i) Construction of the least order completely controllable system realizing the impulse response matrix of (2.1) - i.e., that part affected by the input.

(ii) Construction of the least order completely observable system realizing the impulse response matrix of (2.1) - i.e., that part which affects the output.

By combining (i) and (ii) the least order completely controllable and completely observable subsystem may be found.

Definition 3.1: System (2.1) is reducible (from the input) to order $q \leq n$ and no lower order if it is equivalent to a system of the form

$$\dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix} u$$

$$y = [\hat{C}_1 \quad \hat{C}_2] z \tag{3.20}$$

and the q^{th} order subsystem

$$\dot{z}_1 = \hat{A}_{11} z_1 + \hat{B}_1 u$$

$$y = \hat{C}_1 z_1 \tag{3.21}$$

is completely controllable.

It is clear that the impulse response matrix of (3.21) is equal to that of (3.20) and hence to that of (2.1).

Definition 3.2: System (2.1) is reducible (from the output) to order $q \le n$ and to no lower order if it is equivalent to a system of the form

$$\dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} & 0 \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} u$$

$$y = [\hat{C}_1 \quad 0]\, z \tag{3.22}$$

and the q^{th} order subsystem

$$\dot{z}_1 = \hat{A}_{11} z_1 + \hat{B}_1 u$$

$$y = \hat{C}_1 z_1 \tag{3.23}$$

is completely observable.

Note that in this case the reduced system is zero-state and zero input equivalent (i.e., "equivalent" in the sense of [11]) to the original system.

Only reduction from the input will be treated below, since reduction from the output may be effected in a dual manner.

Conditions under which a system is reducible (non-controllable) will now be explored. The following theorem indicates the importance of the controllability matrix in this context.

<u>Theorem 3.8:</u> System (2.1) is reducible to a q^{th} order totally controllable system if and only if an equivalence transformation $T(t)$ exists such that

$$TQ = \begin{bmatrix} \hat{Q}_1 \\ 0 \end{bmatrix}, \tag{3.24}$$

where $\hat{Q}_1$ has q rows and does not have rank $<q$ on any interval.

It is immediately evident that if $\hat{Q}$ is the controllability matrix of a system of the form (3.20) then

$$\hat{Q} = \begin{bmatrix} \hat{Q}_1 \\ 0 \end{bmatrix}, \tag{3.25}$$

so that the necessity of the conditions of Theorem 3.8 is clear.

To prove sufficiency, it is first noted that Property 3.2 implies

$$\hat{Q}^* = \begin{bmatrix} \hat{Q}_1^* \\ 0 \end{bmatrix},$$

where $\hat{Q}_1^*$ has q rows, so that from (3.19)

$$\begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} \hat{Q}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{d}{dt}\hat{Q}_1 - \hat{Q}_1^* \\ 0 \end{bmatrix}.$$

Since $\hat{Q}_1$ does not have rank $<q$ on any interval, and

$$\hat{A}_{21}\hat{Q}_1 = 0$$

it can be concluded that $\hat{A}_{21} = 0$. Also, since $\hat{B}$ is formed from the first r columns of $\hat{Q}$ it must have the form

$$\hat{B} = \begin{bmatrix} \hat{B}_1 \\ \\ 0 \end{bmatrix},$$

which concludes the proof.

The constraint on $\hat{Q}_1$ in Theorem 3.8 can not in general be relaxed to include reduction to systems that are not totally controllable. It is possible, as shown by Example 3.1, that a system may possess a controllability matrix of the form (3.25) yet not be reducible.

The conditions under which Q admits a transformation satisfying Theorem 3.8 will now be examined.

It is clear that if such a transformation T exists, then its inverse R must satisfy

$$Q = R\hat{Q}$$

and also be an equivalence transformation. The matrix $\hat{Q}$ may be partitioned as

$$\hat{Q} = \begin{bmatrix} \hat{Q}_{11} & \hat{Q}_{12} \\ \\ 0 & 0 \end{bmatrix}$$

where $\hat{Q}_{11}$ is a $q \times (rq)$ matrix not having rank $<q$ on any interval and is the controllability matrix of the reduced system (3.21). Let R be partitioned as $[R_q \vdots R_{n-q}]$ where R_q has q columns and rank q everywhere. Then it is evident that a necessary condition for T to exist is that Q not have rank $<q$ on any interval and that the first rq columns of Q be factorable in the form $R_q\hat{Q}_{11}$. If such factorization is possible then the reducing transformation is given by

$$T^{-1} = [R_q \vdots R_{n-q}] \tag{3.24}$$

where R_{n-q} is any set of $n-q$ continuously differentiable columns making T^{-1} nonsingular.

The above discussion together with Theorem 3.5 implies the following corollary to Theorem 3.8.

<u>Corollary 3.2:</u> System (2.1) is reducible to a totally controllable system of order $q \leq n$ if and only if Q does not have rank $<q$ on any interval, and the first rq columns of Q can be factored in the form $R_q\hat{Q}_{11}$, where R_q has q columns and rank q everywhere and $\hat{Q}_{11}$ does not have rank $<q$ on any interval.

It might appear that the conditions of Corollary 3. 2 are redundant in that the factorability of Q seems to be implied by its rank. That this is not the case is demonstrated by the following example.

Example 3. 2: Consider the second order system

$$\begin{bmatrix} \dot{x}_1 \\ \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ \\ f_2(t) \end{bmatrix} u$$

where

$$f_1(t) = \gamma(t)$$

$$f_2(t) = \gamma(t - 2)$$

and $\gamma(t)$ is as defined in Example 3. 2. For this system,

$$Q = \begin{bmatrix} f_1 & \dot{f}_1 \\ \\ f_2 & \dot{f}_2 \end{bmatrix}$$

has rank 1 almost everywhere on the interval $[-1, 3]$, as does the first column of Q . The system is completely controllable, however, and is thus not reducible on the interval.

A simpler sufficient condition for reducibility applicable for a somewhat smaller class of systems is provided by the following corollary.

<u>Corollary 3.3:</u> If Q has rank q everywhere and q columns of Q also have rank q everywhere then system (2.1) is reducible to a q^{th} order totally controllable system.

Any set of columns of Q having rank q may be used as R_q in (3.24), avoiding the need to factor Q. This method is essentially a generalization of Stubberud's technique [9] for single-input systems. Cases where factorization may be necessary were not considered in [9], however, nor were the precise conditions under which reduction may be performed made clear.

Observe that it is possible for the Q matrix to have rank q everywhere without any q columns having rank q everywhere, as in the following example.

<u>Example 3.3:</u> Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \sin t \\ \sin t \end{bmatrix} u$$

$$y = x_1 .$$

The controllability matrix of this system,

$$Q = \begin{bmatrix} \sin t & \cos t \\ \sin t & \cos t \end{bmatrix}$$

has rank 1 for all t but no single column has this property, so that

Corollary 3.3 is not applicable. It is clear, however, that the first column of Q may be factored as

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin t \ ,$$

so that

$$T^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

will reduce the system to

$$\dot{z}_1 = (\sin t) u$$
$$y = z_1 \quad .$$

One difficulty involved in using the method of reduction associated with Corollaries 3.2 and 3.3 is that the choice of the columns R_{n-q} may not be obvious. Theorem 3.9 below provides a method of reduction in terms of the rows of Q, which has the virtue of providing a simple explicit form for the reducing transformation and the reduced system.

Theorem 3.9: If Q has rank $q < n$ everywhere, and q rows of Q, say Q_1, have this property, then system (2.1) can be reduced by the transformation

$$T = \begin{bmatrix} I_q & 0 \\ -K & I_{n-q} \end{bmatrix}$$

to a totally controllable system of order q where the rows of Q have been reordered so that

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix},$$

and

$$Q_2 = KQ_1 .$$

As in (3.11), K may be expressed as

$$K = Q_2 Q_1^{\dagger} , \qquad (3.25)$$

so that T is an equivalence transformation. It is clear that

$$TQ = \begin{bmatrix} Q_1 \\ 0 \end{bmatrix},$$

which proves that T does indeed reduce the system.

Although (3.25) provides an explicit form for the proportionality matrix K, it is often easier to find K by alternate methods. A method that is quite useful and widely applicable is presented below.

If in addition to the conditions of Theorem 3.9, q columns of Q_1, which will be denoted as Q_{11}, also have rank q everywhere then

$$K = Q_{21}Q_{11}^{-1} , \tag{3.26}$$

where Q_{21} denotes the columns of Q_2 corresponding to the columns of Q_{11}. For this case, simple explicit forms for the matrices of the reduced system can be derived.

Define Q_{11}^* to be that submatrix of Q^* which corresponds to Q_{11}; that is,

$$Q_{11}^* = -A \begin{bmatrix} Q_{11} \\ Q_{12} \end{bmatrix} + Q_{11} .$$

Also, let Q_1^* be the first q rows of Q^*. Then since

$$\hat{Q}^* = \begin{bmatrix} Q_1^* \\ 0 \end{bmatrix}$$

and

$$\hat{Q}_{11} = Q_{11} , \quad \hat{Q}_{11}^* = Q_{11}^*$$

it follows from (3.18) that

$$\hat{A}_{11} = (\dot{Q}_{11} - Q_{11}^{*})Q_{11}^{-1} \; . \tag{3.27}$$

It is obvious that

$$\hat{B}_1 = B_1 \tag{3.28}$$

and

$$\hat{C}_1 = [\, C_1 \; \vdots \; C_2] \begin{bmatrix} I_q \\ Q_{21}Q_{11}^{-1} \end{bmatrix}$$

or

$$\hat{C}_1 = C_1 + C_2 Q_{21} Q_{11}^{-1} \; . \tag{3.29}$$

In addition to giving an explicit form to the reduced system this reduction method has many computational advantages. Any method of system reduction involves finding a transformation of coordinates. The usual reduction procedure is to find the matrices of the reduced system from the appropriate submatrices of the transformed coefficient matrices obtained via (2. 12). This approach, which has been used by Kalman [8] for fixed systems and Stubberud [9] for time-variable single-input single-output systems, requires the inversion and multiplication of $n \times n$ matrices. If relations (3. 26) - (3. 28) are utilized, however, only a single matrix inversion (of a $q \times q$ matrix) and a single matrix multiplication (of $q \times q$ matrices) are required to find $\hat{A}_{11}$. The construction of $\hat{C}_1$ is also simplified and no computation is required for $\hat{B}_1$.

It should be emphasized that this explicit method of system reduction is always applicable for fixed systems, since if Q has rank q it is trivially true that some submatrix of the form Q_{11} also has rank q.

An example of the reduction procedure for a time-variable system follows.

Example 3.4: Consider the third order system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ \cos 2t & -\sin 2t & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \sin t \\ \cos t \\ \sin t \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} .$$

The controllability matrix of this system is

$$Q = \begin{bmatrix} \sin t & -\cos t & -\sin t \\ \cos t & \sin t & -\cos t \\ \sin t & \cos t & -\sin t \end{bmatrix}$$

and its rank is 2 for all t. Also, the submatrix

$$Q_{11} = \begin{bmatrix} \sin t & -\cos t \\ \cos t & \sin t \end{bmatrix}$$

is nonsingular for all t and has as its inverse

$$Q_{11}^{-1} = \begin{bmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{bmatrix} .$$

For this choice of Q_{11},

$$Q_{21} = [\sin t \quad \cos t]$$

and

$$Q_{11}^{*} = \begin{bmatrix} -\cos t & -\sin t \\ \sin t & -\cos t \end{bmatrix} .$$

Therefore it is seen from (3.26) and Theorem 3.9 that

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & \sin 2t & 1 \end{bmatrix}$$

and from (3.27), (3.28) and (3.29) that

$$\hat{A}_{11} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \quad \hat{B}_1 = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}, \quad \hat{C}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

3.6 Fixed Systems

Several of the results of this chapter will now be specialized and applied to the analysis of fixed systems. It will be seen both here

and in subsequent chapters that the more general approach necessitated by the consideration of time-variable systems leads to a clarification and simplification of many fixed system problems.

The reducibility criteria for fixed systems simplify considerably, as evidenced by the following theorem

Theorem 3.10: A fixed system of the form (2.1) is reducible from the input to a q^{th}-order completely controllable system, if and only if Q_c has rank q.

The proof of this theorem follows directly from Theorem 3.8 and the fact that if Q_c is a constant matrix of rank q, there always exists a constant non-singular matrix T such that

$$TQ_c = \begin{bmatrix} \bar{Q}_{c1} \\ \\ 0 \end{bmatrix}$$

where $\bar{Q}_{c1}$ has q rows and rank q.

If the system state-variables are so ordered that the first q rows of Q_c have rank q, then

$$T = \begin{bmatrix} I_q & 0 \\ \\ -K & I_{n-q} \end{bmatrix}$$

where K is given by equation (3.26), and the reduced system by equations (3.28), (3.29), and

$$\hat{A}_{11} = Q_{11}^{*} Q_{11}^{-1} \qquad (3.30)$$

Similarly, we have:

Theorem 3.11: A fixed system of the form (2.1) is reducible from the output to a q^{th}-order completely observable system if and only if Q_c has rank q .

An important advantage of the criteria for reducibility given by Theorems 3.10 and 3.11, as well as the reduction procedure presented, is that they require operations solely on the controllability and observability matrices. Furthermore, as will now be shown, they lead to an explicit measure of the degree of the least order system obtainable by reduction from both the input and the output (i.e., the least order system realizing the system's impulse response matrix).

Theorem 3.12: Let Q_c and Q_0 be the controllability and observability matrices of a fixed system of the form (2.1), then:

(1) The system is irreducible [8] (completely controllable and completely observable) if and only if the matrix $Q_0'Q_c$ has rank n .

(2) If $Q_0'Q_c$ has rank $q < n$, then (2.1) is zero-state equivalent to an irreducible system of order q ,

$$\dot{v} = \bar{A}v + \bar{B}u$$
$$y = \bar{C}v . \tag{3.31}$$

Furthermore,

$$Q_0'Q_c = \begin{bmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \cdot \\ \cdot \\ \cdot \\ \bar{C}\bar{A}^{q-1} \\ \vdots \\ \bar{C}\bar{A}^{n-1} \end{bmatrix} [\bar{B} \vdots \bar{A}\bar{B} \vdots \ldots \vdots \bar{A}^{q-1}\bar{B} \vdots \ldots \vdots \bar{A}^{n-1}\bar{B}] \ . \tag{3.32}$$

The proof of part (1) is easily established. If $Q_0'Q_c$ has rank n, then both Q_0 and Q_c must have rank n which in turn implies the system is irreducible. If $Q_0'Q_c$ has rank $<n$ then either Q_0 or Q_c has rank $<n$, and the system is reducible either from the input or the output.

Part (2) is established by first transforming system (2.1) to the equivalent form

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & 0 & \tilde{A}_{13} & \tilde{A}_{14} \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} \\ 0 & 0 & \tilde{A}_{33} & \tilde{A}_{34} \\ 0 & 0 & \tilde{A}_{43} & \tilde{A}_{44} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ 0 \\ 0 \end{bmatrix} u \tag{3.33}$$

$$y = [\tilde{C}_1 \quad 0 \quad \tilde{C}_3 \quad \tilde{C}_4] z$$

where z_1 is a q-vector and the subsystem

$$\dot{z}_1 = \tilde{A}_{11} z_1 + \tilde{B}_1 u$$

$$y = \tilde{C}_1 z_1$$

is completely controllable and observable. The procedure for constructing this equivalent system follows.

Let $q_c \le n$ be the rank of Q_c, then by Theorem 3.10 there exists a constant nonsingular transformation T_c such that if $w = T_c x$,

$$\dot{w} = \begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix} u$$

$$y = [\hat{C}_1 \quad \hat{C}_2 \,]\, w \;,$$

and

$$\hat{Q}_c = T_c Q_c = \begin{bmatrix} Q_{c1} \\ 0 \end{bmatrix} ,$$

where w_1 and $\hat{Q}_{c1}$ have q_c rows and $\hat{Q}_{c1}$ has rank q_c. If $\hat{Q}_0$ is partitioned as

$$\hat{Q}_0' = [\,\hat{Q}_{01}' \,\vdots\, \hat{Q}_{02}'] = Q_0' T_c^{-1}$$

where $\hat{Q}_{01}'$ has q_c columns, then clearly

$$Q_0' Q_c = \hat{Q}_{01}' \hat{Q}_{c1} \;. \tag{3.34}$$

Since $\hat{Q}_{c1}$ has rank q_c,

$$\hat{Q}_{01}' = Q_0' Q_c \hat{Q}_{c1}' (\hat{Q}_{c1} \hat{Q}_{c1}')^{-1} = Q_0' Q_c \hat{Q}_{c1}^{\dagger}$$

which implies that the rank of $\hat{Q}_{01}$ is $\leq q$. Equation (3.34), however, implies that $\hat{Q}_{01}$ has rank $\geq q$. Therefore, $\hat{Q}_{01}$ must have rank q. It follows then from Theorem 3.11 that there exists a constant nonsingular matrix T_0 such that if

$$z = \begin{bmatrix} T_0 & 0 \\ 0 & I_{n-q_c} \end{bmatrix} w ,$$

then (2.1) is equivalent to a system of the form (3.33). Thus,

$$\begin{aligned} \tilde{Q}_0' &= [\hat{Q}_{01}' \vdots \hat{Q}_{02}'] \begin{bmatrix} T_0^{-1} & 0 \\ 0 & I_{n-q_c} \end{bmatrix} \\ &= [\tilde{Q}_{01}' \vdots 0 \vdots \tilde{Q}_{03}' \vdots \tilde{Q}_{04}'] \end{aligned}$$

where $\tilde{Q}_{01}'$ has q columns and rank q.

If $\tilde{Q}_c$ is partitioned as

$$\tilde{Q}_c = \begin{bmatrix} \tilde{Q}_{c1} \\ \tilde{Q}_{c2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} T_0 & 0 \\ 0 & I_{n-q_c} \end{bmatrix} \begin{bmatrix} \hat{Q}_{c1} \\ 0 \end{bmatrix}$$

where $\tilde{Q}_{c1}$ has q rows, then

$$Q_0'Q_c = \tilde{Q}_{01}'\tilde{Q}_{c1}$$

and $\tilde{Q}_{c1}$ must have rank q . Making the identifications $v = z_1$, $\bar{A} = \tilde{A}_{11}$, $\bar{B} = \tilde{B}_1$, $\bar{C} = \tilde{C}_1$ and

$$\tilde{Q}_{c1} = [\bar{B} \vdots \bar{A}\bar{B} \vdots \ldots \vdots \bar{A}^{n-1}\bar{B}]$$

$$\tilde{Q}_{01} = [\bar{C}' \vdots \bar{A}'\bar{C}' \vdots \ldots \vdots (\bar{A}^{n-1})'\bar{C}']$$

completes the proof.

<u>Example 3.5</u>: Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 2 & -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix} u$$

$$y = [1 \quad 1 \quad 0 \quad -1]x .$$

For this system

$$Q_c = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & -1 \\ 2 & 1 & 0 & -3 \\ 2 & 1 & 0 & -1 \end{bmatrix}$$

and

$$Q_0' = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & -3 & 4 & -1 \\ 1 & 1 & 0 & -3 \\ 1 & -3 & 4 & -1 \end{bmatrix} .$$

Although both Q_c and Q_0 have rank 2, their product

$$Q_0'Q_c = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} ,$$

has rank 1, so that the system is reducible to a first order system. To find the reduced system we first reduce the given system from the input. Note that the submatrix

$$Q_{11} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

of Q_c is nonsingular. With this choice of Q_{11},

$$Q_{11}^* = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

so that from (3.30),

$$\hat{A}_{11} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

and

$$\hat{B}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad C_1 = [1 \quad 0] .$$

By inspection, therefore, the final reduced system is

$$\dot{v} = v + u$$

$$y = v \quad .$$

It will be shown in Chapter 4, that the form of the product $Q_0'Q_c$ given by (3. 32) leads to a new direct method for constructing a least order system zero-state equivalent to (2. 1). This procedure avoids the successive transformation required by Theorems 3. 10 and 3. 11.

3. 7 Analytic Systems

Most of the results obtained in the previous section for fixed systems generalize directly for systems with analytic coefficient matrices, as will now be demonstrated.

Theorem 3. 13: An analytic system of the form (2. 1) is reducible from the input to a q^{th}-order completely controllable system, if and only if Q_c has rank q .

The necessity of the reducibility criteria is obvious. To prove sufficiency note that if Q_c (and thus W_c) has rank q, Corollary 3.2 implies that exactly q rows of $X^{-1}(t)B(t)$ are linearly independent. Therefore, there must exist a constant non-singular matrix T_1 such that

$$T_1 W_c = \begin{bmatrix} \overline{W}_c \\ 0 \end{bmatrix}$$

where $\overline{W}_c$ has q rows and rank q. Clearly, $T = T_1 X^{-1}$ is an equivalence transformation and from (3.15),

$$TQ_c = \begin{bmatrix} \overline{W}_c \\ 0 \end{bmatrix}.$$

Thus, Theorem 3.8 implies that the system is reducible to a q^{th} order completely controllable system.

In the case that q rows of Q_c have rank q for <u>all</u> t, the explicit form of the reducing transformation given in Theorem 3.9 may be utilized. If, as may happen, Q_c does not have maximal rank for some t, the factorization procedure may be necessary to find the transformation.

In a dual manner to the above it may also be shown that:

<u>Theorem 3.14</u>: An analytic system of the form (2.1) is reducible from the output to a q^{th}-order completely observable system if and only if Q_o has rank q.

Since the proof of Theorem 3.12 depended solely on Theorems 3.10 and 3.11 it is clear that a completely parallel proof based on Theorems 3.13 and 3.14 may be given for the following.

Theorem 3.15: Let Q_c and Q_0 be the controllability and observability matrices of an analytic system of the form (2.1), then:

(1) The system is irreducible if and only if the matrix $Q_0'Q_c$ has rank n.

(2) If $Q_0'Q_c$ has rank $q < n$ then (2.1) is zero-state equivalent to an irreducible system of order q,

$$\begin{aligned} \dot{v} &= \bar{A}v + \bar{B}u \\ y &= \bar{C}v \end{aligned} \qquad (3.34)$$

Furthermore,

$$Q_0'Q_c = \begin{bmatrix} \bar{S}_0' \\ \bar{S}_1' \\ \cdot \\ \cdot \\ \cdot \\ \bar{S}_{q-1}' \\ \cdot \\ \cdot \\ \cdot \\ \bar{S}_{n-1} \end{bmatrix} [\bar{P}_0 \vdots \bar{P}_1 \vdots \cdots \vdots \bar{P}_{q-1} \vdots \cdots \vdots \bar{P}_{n-1}] \qquad (3.35)$$

In Chapter 4, it will be shown that an extremely efficient method for reducing analytic systems follows from the form of $Q_0'Q_c$ given by (3.35), as well as several important results pertaining to the representation of such systems.

References

[1] A. R. Stubberud, "A Controllability Criterion for a Class of Linear Systems," IEEE Trans. on Appl. and Ind., vol. 68, pp. 411-413; November, 1964.

[2] A. Chang, "An Algebraic Characterization of Controllability," IEEE Trans. on Automatic Control, vol. AC-10, pp. 112-113; January, 1965.

[3] L. M. Silverman and H. E. Meadows, "Controllability and Time-Variable Unilateral Networks," IEEE Trans. on Circuit Theory, vol. CT-12, pp. 308-313; Sept. 1965.

[4] L. M. Silverman and H. E. Meadows, "Degrees of Controllability in Time-Variable Linear Systems," Proceedings of the National Electronics Conf., vol. 21, pp. 689-693; 1965.

[5] W. Hurewicz, Lectures on Ordinary Differential Equations, M. I. T. Press, Cambridge, Mass., 1958.

[6] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, The Mathematical Theory of Optimal Processes, Interscience, New York; 1962.

[7] E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill Book Co., Inc., New York, N. Y.; 1955.

[8] R. E. Kalman, "Mathematical Description of Linear Dynamical Systems," J. SIAM Control, vol. 1, pp. 152-192; 1963.

[9] A. R. Stubberud, "Reducibility of Linear Systems," IEEE Trans. on Appl. and Ind., vol. 67, pp. 214-219; July, 1963.

[10] C. A. Desoer, "The Minimal Realization of Impulse Response Matrices," ERL Tech Memorandum M-128, University of California, Berkeley; September, 1965.

[11] L. A. Zadeh and C. A. Desoer, Linear System Theory, McGraw-Hill Book Co., Inc., New York, N. Y.; 1963.

CHAPTER 4

SYSTEM EQUIVALENCE AND CANONICAL REPRESENTATION*

4.1 Introduction

A new degree of controllability and observability, stronger than those previously defined, will be introduced in this chapter. It will be shown that the state of a system possessing these properties can be controlled and observed "instantaneously." The motivation for defining these properties (to be called uniform controllability and uniform observability), arises from the role they play in the study of equivalent systems. There are many problems in the analysis, synthesis and control of linear systems that can be solved most effectively by transforming a given system to an appropriate equivalent representation. For example, when simulating a time-variable system on an analogue computer, it is desirable to minimize the number of variable components (e.g., multipliers) required. There are several well known canonical structures for single-input single-output systems which require no more than $2n$ multipliers [1, 2, 3]. It is therefore desirable to know the conditions under which a system has an equivalent canonical representation and to be able to construct such representations when they exist.

For fixed systems, conditions for a system to be equivalent to several particular canonical forms are well known [1], but the

---

*Parts of this chapter have appeared in two papers by the author and H. E. Meadows [18, 19].

techniques for transforming to these forms are often complex or ad-hoc methods [4, 5, 6]. There are few previous results for time-variable systems in this area.

We will present below a general approach to the problem of system equivalence for both fixed and time-variable systems. For the class of uniformly controllable and observable systems (which includes fixed completely controllable and observable systems), a necessary and sufficient condition for equivalence of two systems will be given which requires no knowledge of their solutions. In addition, an explicit form for the transformation between equivalent systems is found. New insight into the problem of system reduction is gained from this study, and a method for directly finding the class of minimal order systems zero-state equivalent to a prescribed system will be given. An interesting and potentially useful byproduct is a necessary and sufficient condition for an analytic system to be zero-state time-invariant and a method for finding an equivalent fixed system if such is the case.

Two important canonical systems will be studied in detail- the input-output differential equation form, and the "phase-variable" canonical form. It will be shown that uniform observability and uniform controllability, respectively, are necessary and sufficient for the existence of equivalent systems in these forms. Explicit methods for constructing canonical equivalents will also be given.

4.2 Uniform Controllability and Uniform Observability

In the previous chapter conditions for complete and total controllability were given in terms of the matrix Q_c. As shown, it is not necessary for this matrix to have maximal rank at all points to insure either of these types of controllability. A point to be emphasized in the present chapter is that for many problems of system equivalence the controllability matrix must have rank n for all t.

Thus we define the following new degree of controllability.

Definition 4.1: System (2.1) is said to be uniformly controllable on an interval $[t_0, t_1]$ if the matrix Q_c has rank n for all $t \in [t_0, t_1]$.

A new degree of observability is defined similarly:

Definition 4.2: System (2.1) is said to be uniformly observable on an interval $[t_0, t_1]$ if the matrix Q_0 has rank n for all $t \in [t_0, t_1]$.

An interesting interpretation of uniform controllability can be made which shows how this criterion relates to the more familiar types of controllability which arise in optimum control problems. If a system is totally controllable, then by definition the state of the system may be transferred to any desired value in an arbitrarily short interval of time by application of some input. It will now be shown that if a system is uniformly controllable, it is even possible to perform the state transition instantaneously. Furthermore, an explicit input in terms of the controllability matrix will be given which effects the transition.

It is first necessary to examine the response of a time-variable system to impulse functions and their derivatives. In this discussion we will rely heavily on the development of Zadeh and Desoer [3]. Since the response to an impulse is discontinuous in general, it is necessary to distinguish the value of the state prior to and after the application of an impulse. The notation t^- will denote the left hand side of any discontinuity.

Suppose the state of system (2.1) is zero at time t_0^- and $u(t) = \delta(t - t_0)\alpha_0$ where α_0 is an r-vector of arbitrary constants, then

$$x(t) = \int_{t_0^-}^{t} \Phi(t,\tau)B(\tau)\delta(\tau - t_0)\alpha_0 d\tau, \qquad t > t_0 .$$

By the "sifting property" of the impulse function

$$x(t) = \Phi(t,\tau)B(\tau)\alpha_0 \Big|_{\tau = t_0}, \qquad t > t_0$$

or

$$x(t) = \Phi(t,t_0)B(t_0)\alpha_0, \qquad t > t_0 .$$

Thus, it may be said that the state at time t_0 has changed instantaneously from zero to

$$x(t_0) = B(t_0)\alpha_0 = P_0(t_0)\alpha_0 . \qquad (4.1)$$

By generalizing the argument of Zadeh and Desoer ([3] page 496) to time-variable systems and utilizing (3. 7) it can be seen that if $u(t) = \delta^{(k)}(t - t_0)\alpha_k$ is applied to system (2. 1) the state will "jump" to the value

$$x(t_0) = P_k(t_0)\alpha_k , \tag{4. 2}$$

where $P_k(t)$ is defined by the recursion formula (3. 2).

With this development, it can now be shown that if system (2. 1) is uniformly controllable, its state can be changed instantaneously to any desired value with an input of the form

$$u(t) = \sum_{k=0}^{n-1} \delta^{(k)}(t - t_0)\alpha_k . \tag{4. 3}$$

Let x_d be the desired state at t_0 and x_i be the initial state at t_0^- ; then it is clear that if $u(t)$ is given by (4. 3)

$$(x_d - x_i) = \sum_{k=0}^{n-1} P_k(t_0)\alpha_k . \tag{4. 4}$$

If the nr vector α is defined as

$$\alpha = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{bmatrix},$$

then (4.4) can be rewritten as

$$(x_d - x_i) = Q_c(t_0)\alpha . \qquad (4.5)$$

Equation (4.5) represents n equations in rr unknowns, and if $Q_c(t_0)$ has rank n, a solution (non-unique) for α exists. An explicit solution is given by

$$\alpha = Q_c'(t_0)[Q_c(t_0)Q_c'(t_0)]^{-1}(x_d - x_i)$$

$$= Q_c^{\dagger}(t_0)(x_d - x_i).$$

If the matrix $\Delta(t)$ is defined as

$$\Delta(t) = [\delta(t)I_r \,\vdots\, \delta^{(1)}(t)I_r \,\vdots\, \ldots \,\vdots\, \delta^{(n-1)}(t)I_r],$$

then an input which changes the system state from x_i to x_d at t_0 is

$$u(t) = \Delta(t - t_0)Q_c^{\dagger}(t_0)(x_d - x_i) . \qquad (4.6)$$

For a single-input system the above solution is unique and (4.6) reduces to

$$u(t) = \Delta(t - t_0)Q_c^{-1}(t_0)(x_d - x_i)$$

where

$$\Delta(t) = [\,\delta(t) \vdots \delta^{(1)}(t) \vdots \ldots \vdots \delta^{(n-1)}(t)\,] .$$

It is interesting to observe that it may be possible to change the state instantaneously at t_0 even if $Q_c(t_0)$ does not have rank n. The reason for this is that appending further terms of the sequence of P_k's to Q_c can increase its rank at a point even though it cannot increase its rank over an interval (Property 3. 2). A simple example which demonstrates this is the first order system

$$\dot{x} = tu$$

For this system $Q_c = t$, and has rank 0 at $t = 0$. However, $P_1 = 1$ and the input

$$u(t) = \delta^{(1)}(t)(x_d - x_i)$$

will change the system state from x_i to x_d at $t = 0$.

A dual result for uniform observability may also be established.

Let $u = 0$ for convenience, and differentiate the output of (2. 1) n-1 times. Then it is clear that

$$Y(t) \triangleq \begin{bmatrix} y(t) \\ y^{(1)}(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix} = Q_0'(t)x(t) \quad . \tag{4.7}$$

If $Q_0(t_0)$ has rank n, then (4.7) can be solved uniquely for $x(t_0)$, with the solution given explicitly as

$$x(t_0) = (Q_0(t_0)Q_0'(t_0))^{-1}Q_0(t_0)Y(t_0)$$

or

$$x(t_0) = (Q_0'(t_0))^{\dagger}\, Y(t_0) \, . \tag{4.8}$$

That is, if system (2.1) is uniformly observable, the state of the system at any time may be determined instantaneously from observations of the system output and its derivatives.

4.3 Equivalent Systems

Consider now the class of all uniformly observable and controllable systems of order n. This class will be denoted by U_n. If a system of the form (2.1) is a member of U_n, the notation

$$(A, B, C) \in U_n$$

will be used.

If (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ are equivalent systems of order n, where

$$\bar{A} = TAT^{-1} + \dot{T}T^{-1} \tag{4.9}$$

$$\bar{B} = TB \tag{4.10}$$

$$\bar{C} = CT^{-1} \tag{4.11}$$

the relationship between the two systems will be represented symbolically as

$$(A, B, C) \xrightarrow{T} (\bar{A}, \bar{B}, \bar{C}) \; . \tag{4.12}$$

If (4.12) holds, then clearly

$$(\bar{A}, \bar{B}, \bar{C}) \xrightarrow{T^{-1}} (A, B, C) \; . \tag{4.13}$$

Two equivalent systems (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ will be said to belong to the same equivalence class. The primary goal of this section is the characterization of equivalence classes of uniformly observable and controllable systems. It will be shown that for each such class there is a unique invariant matrix, knowledge of which suffices to generate all members of the class. We will use this property to derive several important results including:

(1) A method for directly finding the class of all systems of minimal order which are zero-state equivalent to a given system.

(2) A necessary and sufficient condition for an analytic system to be zero-state time-invariant (i. e., to have a stationary impulse response matrix), and a method for finding an equivalent fixed system when such exists.

Recall that if Q_c and Q_0 are the controllability and observability matrices of (A, B, C) and $\bar{Q}_c$ and $\bar{Q}_0$ are the corresponding matrices for $(\bar{A}, \bar{B}, \bar{C})$, the relationship

$$(A, B, C) \xrightarrow{T} (\bar{A}, \bar{B}, \bar{C}) ,$$

implies that

$$\bar{Q}_c = TQ_c \tag{4.14}$$

$$\bar{Q}_0' = Q_0'T^{-1} . \tag{4.15}$$

Clearly, if (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ are equivalent, then $(A, B, C) \in U_n$ if and only if $(\bar{A}, \bar{B}, \bar{C}) \in U_n$. Equations (4. 14) and (4. 15) also directly imply the following property of systems belonging to U_n.

Property 4. 1: If $(A, B, C) \in U_n$ and $(\bar{A}, \bar{B}, \bar{C}) \in U_n$ are equivalent where

$$(A, B, C) \xrightarrow{T} (\bar{A}, \bar{B}, \bar{C})$$

then

$$T = \bar{Q}_c Q_c^{\dagger} = (Q_0 \bar{Q}_0^{\dagger})' \tag{4.16}$$

or equivalently,

$$\bar{Q}_0{}' \bar{Q}_c = Q_0' Q_c . \tag{4.17}$$

The explicit forms of the transformation between equivalent systems given by (4.16) will be extremely useful in the construction of canonical representations for time-variable systems. As a preliminary demonstration of the utility of (4.16), it will be used to derive an explicit expression for the impulse response of a fixed single-input system (A, b, C). For simplicity, it is assumed that A has distinct eigenvalues λ_i. Under this condition it is well known [1] that if (A, b, C) is completely controllable it is equivalent to $(\bar{A}, \bar{b}, \bar{C})$, where

$$\bar{A} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} .$$

The controllability matrix of $(\bar{A}, \bar{b}, \bar{C})$ is seen by inspection to be the

Vandermonde matrix of A ,

$$V = \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

It follows from Property 4.1, therefore, that

$$(A,\ b,\ C) \xrightarrow{T} (\bar{A},\ \bar{b},\ \bar{C}),$$

where

$$T = VQ^{-1}$$

and Q is the fixed form of the controllability matrix of (A, b, C),

$$Q = [\,b \,\vdots\, Ab \,\vdots\, \ldots \,\vdots\, A^{n-1}b\,] .$$

The impulse response matrix of (A, b, C) is thus given by

$$H(t) = CQV^{-1}E(t) , \tag{4.18}$$

where

$$E(t) = \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix} .$$

Since there are many efficient methods for inverting Vandermonde matrices [7, 8], equation (4.18) would appear to be a computationally advantageous way of calculating the impulse response of a fixed system.

A result which follows immediately from Property 4.1 is a theorem stated by Kalman [1] and recently established by Youla [9].

Theorem 4.1: Let $(A, B, C) \in U_n$ and $(\bar{A}, \bar{B}, \bar{C}) \in U_n$ be fixed equivalent systems. Then

$$(A, B, C) \xrightarrow{T} (\bar{A}, \bar{B}, \bar{C}) ,$$

where T is a constant matrix.

To prove this theorem, merely note that Q_c and $\ddot{Q}_c$ must be fixed, and apply Property 4.1.

Although Property 4.1 gives a method for finding the form of the transformation between equivalent systems, we do not as yet have a criterion for determining whether two given systems are equivalent. Before considering this problem, it is necessary to introduce two new system matrices.

Define,

$$Q_c^+ = [P_0 \vdots P_1 \vdots \ldots \vdots P_n]$$

where

$$P_k = -AP_{k-1} + \dot{P}_{k-1} \; ; \quad P_0 = B \, ,$$

and define

$$Q_0^+ = [S_0 \vdots S_1 \vdots \ldots \vdots S_n]$$

where

$$S_k = A'S_{k-1} + \dot{S}_{k-1} \; ; \quad S_0 = C' \, .$$

It is clear that Q_c and Q_c^* are proper submatrices of Q_c^+, and that Q_0 and Q_0^* are similarly proper submatrices of Q_0^+. We will refer to Q_c^+ and Q_0^+ as the <u>identification matrices</u> of (A, B, C). The choice of this name is made clear by the following property.

<u>Property 4.2</u>: The matrices Q_0^+ and B (or Q_c^+ and C) are sufficient to uniquely specify (A, B, C) $\epsilon \, U_n$.

Property 4.2 was essentially proved in Chapter 3 since

$$A = (\dot{Q}_c - Q_c^*)Q_c^\dagger \qquad (4.19)$$

$$= [(Q_0^* - \dot{Q}_0)Q_0^\dagger]' \, , \qquad (4.20)$$

and

$$B = P_0$$

$$C = S_0' \,.$$

Let F_c be an $n \times n$ submatrix of Q_c having rank n for all t (if such exists) and let F_c^* be the corresponding submatrix of Q_c^*, that is

$$F_c^* = -AF_c + \dot{F}_c \,.$$

Then, equation (4.19) may be simplified to

$$A = (\dot{F}_c - F_c^*)F_c^{-1} \,. \tag{4.21}$$

Similarly, let F_0 be an $n \times n$ submatrix of Q_0' having rank n for all t and F_0^* the corresponding submatrix of $(Q_0^*)'$; then

$$A = F_0^{-1}(F_0^* - \dot{F}_0) \,. \tag{4.22}$$

It is now possible to establish a necessary and sufficient condition for two uniformly observable and controllable systems to be equivalent. This condition, which requires no knowledge of the system responses, does not appear to have been shown previously, even for fixed systems.

<u>Theorem 4.2:</u> Let (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ belong to U_n, then the two systems are equivalent if and only if

$$(Q_0^+)'Q_c^+ = (\bar{Q}_0^+)'\bar{Q}_c^+ . \tag{4.23}$$

That (4.23) is necessary for equivalence follows directly from Property (4.1).

To prove sufficiency, observe that if (4.23) holds then certainly

$$Q_0'Q_c^+ = \bar{Q}_0'\bar{Q}_c^+ \quad , \tag{4.24}$$

since this equation is simply the first mn rows of (4.23). By assumption, Q_0 and $\bar{Q}_0$ have rank n for all t, so that

$$\bar{Q}_c^+ = TQ_c^+$$

where

$$T = (\bar{Q}_0')^\dagger Q_0'$$

and T has rank n for all t. Since Q_c has rank n for all t it must also be true that

$$(\bar{Q}_0^+)' = (Q_0^+)'T^{-1} .$$

The form of T implies that it is continuously differentiable and thus an equivalence transformation. Consequently, there exist $(\hat{A}, \hat{B}, \hat{C}) \in U_n$ such that

$$(A, B, C) \xrightarrow{T} (\hat{A}, \hat{B}, \hat{C}) .$$

Let $\hat{Q}_c^+$ and $\hat{Q}_0^+$ be the identification matrices of $(\hat{A}, \hat{B}, \hat{C})$; then (4.14) implies that

$$\hat{Q}_c^+ = TQ_c^+ = \overline{Q}_c^+$$

Thus by Property 4.2,

$$(\hat{A}, \hat{B}, \hat{C}) = (\overline{A}, \overline{B}, \overline{C}) ,$$

and $(\overline{A}, \overline{B}, \overline{C})$ is equivalent to (A, B, C), which completes the proof.

It should be observed that it suffices for equivalence that either

$$(Q_0^+)'Q_c = (\overline{Q}_0^+)'\overline{Q}_c$$

or

$$Q_0'Q_c^+ = \overline{Q}_0'\overline{Q}_c^+ .$$

The matrix $\Upsilon \triangleq (Q_0^+)'Q_c^+$ is thus seen to be an invariant quantity which essentially characterizes an equivalence class of systems. It will now be shown that knowledge of Υ alone (and not its factors) serves to determine its equivalence class.

Let Υ be given for some equivalence class of systems and let $(A, B, C) \in U_n$ be any member of the class. All submatrices of Υ are invariant under transformation of coordinates so that we can identify the following useful submatrices:

(1) $\Gamma = Q_0'Q_c$

is the matrix formed from the first mn rows and the first rn columns of Υ;

(2) $\Gamma^* = (Q_0^*)'Q_c$

is the matrix formed from the last mn rows and the first rn columns of Υ.

Suppose that Γ contains an $n \times n$ submatrix F having rank n for all t, then it is clear that F is of the form

$$F = F_0F_c \tag{4.25}$$

where F_0 and F_c are $n \times n$ submatrices of Q_0' and Q_c respectively.

Let F^* be defined as the corresponding submatrix of Γ^*, that is

$$F^* = F_0^* F_c \tag{4.26}$$

where

$$F_0^* = F_0 A + \dot{F}_0 .$$

Also, let F_1 be the matrix composed of those columns of the first m rows of Γ which correspond to F, that is

$$F_1 = CF_c . \tag{4.27}$$

Similarly, let F_2 be the matrix composed of those rows of the first r columns of Γ which correspond to the rows of F, that is

$$F_2 = F_0 B . \tag{4.28}$$

Observe that the matrices F, F^*, F_1 and F_2 are all submatrices of Υ and can be determined <u>without</u> knowing the matrices of any particular system (A, B, C). What we will now show is that a member of the equivalence class associated with Υ can be determined directly in ..ns of the above submatrices.

<u>Theorem 4.3:</u> Let Υ be given for an equivalence class of systems in U_n, and let F, F^*, F_1 and F_2 be defined as above. Then, the system

$(\bar{A}, \bar{B}, \bar{C})$ is a member of the equivalence class where

$$\bar{A} = F^* F^{-1}$$

$$\bar{B} = F_2$$

$$\bar{C} = F_1 F^{-1} \quad . \tag{4.29}$$

Let (A, B, C) be an arbitrary member of the equivalence class. Since F has rank n for all t then both F_0 and F_c must have rank n for all t . Consider the transformation of coordinates

$$(A, B, C) \xrightarrow{F_0} (\hat{A}, \hat{B}, \hat{C}) \quad .$$

Under this transformation,

$$\hat{F}_0 = F_0 F_0^{-1} = I_n \ ,$$

$$\hat{F}_c = F_0 F_c \ ,$$

and

$$\hat{F}_0^* = F_0^* F_0^{-1} \ ,$$

so that from (4.22)

$$\hat{A} = \hat{F}_0^{-1}(\hat{F}_0^* - \frac{d}{dt}\hat{F}_0)$$

$$= \hat{F}_0^* = F_0^* F_0^{-1} .$$

Also,

$$\hat{B} = F_0 B$$

and

$$\hat{C} = C F_0^{-1} .$$

But from equations (4. 25) - (4. 28) ,

$$F_0 = F F_c^{-1} ,$$

$$F_0^* = F^* F_c^{-1} ,$$

$$C = F_1 F_c^{-1} ,$$

and

$$B = F_0^{-1} F_2 ,$$

so that

$$\hat{A} = F^* F_c^{-1} F_c F^{-1} = \bar{A}$$

$$\hat{B} = F_0 F_0^{-1} F_2 = \bar{B}$$

and

$$\hat{C} = F_1 F_c^{-1} F_0^{-1} = \bar{C} \ .$$

Thus,

$$(A,\ B,\ C) \xrightarrow{F_0} (\bar{A},\ \bar{B},\ \bar{C})$$

and the theorem is established.

The fact that an equivalence class of systems can be determined from its Υ matrix will now be utilized to derive a new and efficient method of system reduction.

Let (A, B, C) be a fixed system of order n with r inputs and m outputs and define Υ to be the product of its identification matrices Q_0^+ and Q_c^+. If Υ has rank $q < n$, let $\bar{\Upsilon}$ be the submatrix formed from the first $m(q+1)$ rows and the first $r(q+1)$ columns of Υ. It follows then from Theorem 3.12 that there exists a q^{th}-order completely controllable and observable system $(\hat{A}, \hat{B}, \hat{C})$, zero-state equivalent to (A, B, C). Furthermore, if $\hat{Q}_0^+$ and $\hat{Q}_c^+$ are the identification matrices of such a system then

$$\bar{\Upsilon} = (\hat{Q}_0^+)'\hat{Q}_c^+ \ . \tag{4.30}$$

It is clear that Theorem 4.3 may be applied to $\bar{\Upsilon}$ to directly construct a system equivalent to $(\hat{A}, \hat{B}, \hat{C})$ (and thus zero-state equivalent to (A, B, C)). If $\bar{F}$, $\bar{F}^*$, $\bar{F}_1$ and $\bar{F}_2$ are defined as in

Theorem 4.3, as submatrices of $\bar{\Upsilon}$, then the following theorem holds.

Theorem 4.4: If (A, B, C) is a time-invariant system and Υ has rank $q \leq n$ then (A, B, C) is zero-state equivalent to the q^{th} order completely controllable and observable system $(\bar{A}, \bar{B}, \bar{C})$ where

$$\bar{A} = \bar{F}^* \bar{F}^{-1}$$

$$\bar{B} = \bar{F}_2$$

$$\bar{C} = \bar{F}_1 \bar{F}^{-1} . \tag{4.31}$$

There are several advantages of this method of reduction both in comparison with previously published techniques [1, 10] and with that presented in Chapter 3. Most importantly, reduction from the input and the output is performed simultaneously with a resultant saving in computation. Furthermore, it yields a simple explicit form for the reduced system.

To compare this method with that of Chapter 3 consider the system of Example 3.5.

Recall that for this system

$$Q_0' Q_c = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} .$$

Clearly, $Q_0'Q_c$ (and thus Υ) has rank 1 , and

$$\overline{F} = \overline{F}^* = \overline{F}_1 = \overline{F}_2 = 1 .$$

Therefore,

$$\overline{A} = \overline{B} = \overline{C} = 1 .$$

Notice, that no matrix inversions are required by this method whereas by the previous method it was necessary to invert a 2×2 matrix to find the reduced system. It will generally be true that fewer and lower order matrix inversions will re required by the present reduction procedure.

An obvious generalization of Theorem 4. 4 to analytic time-variable systems is:

Theorem 4. 5: If (A, B, C) is an analytic system and Υ has rank $q \le n$ for all t then (A, B, C) is zero-state equivalent to the q^{th} order uniformly controllable and observable system $(\overline{A}, \overline{B}, \overline{C})$ where

$$\overline{A} = \overline{F}^*\overline{F}^{-1}$$

$$\overline{B} = \overline{F}_2$$

$$\overline{C} = \overline{F}_1\overline{F}^{-1} , \qquad (4.32)$$

and $\overline{F}^*$, $\overline{F}_1$ and $\overline{F}_2$ are defined as in Theorem 4.3 as submatrices of $\overline{\Upsilon}$.

It should be noted that the restriction to analytic systems in Theorem 4.5 is a necessary one as the following example demonstrates.

Example 4.1: Consider the first order system

$$\dot{x} = g(t)u$$

$$y = f(t)x$$

where

$$g(t) = \gamma(t-1)$$

$$f(t) = \gamma(t-3)$$

and $\gamma(t)$ is as defined in Example 3.1. It is clear that $\Upsilon \equiv 0$ for t. However, the system is not zero-state equivalent to a zero-order system since its impulse response

$$h(t, \tau) = \gamma(t-3)\gamma(\tau-1) \quad , \quad t \geq \tau$$

is non zero for $\tau \in [0, 2]$ and $t \in [2, 4]$.

An interesting and potentially useful byproduct of Theorems 4.3 and 4.5 is a test to determine if a system is zero-state time-

invariant (i.e., has a stationary impulse response matrix), and a method for finding a fixed zero-state equivalent for such a system. This effectively provides a method for solving a class of time-variable systems in closed form.

Theorem 4.6: Let (A, B, C) be an n^{th} order time-variable system. Then,

(1) (A, B, C) is equivalent to a fixed system if and only if Υ is a constant matrix of rank n.

(2) If (A, B, C) is an analytic system, it is zero-state equivalent to a fixed system if and only if Υ has constant rank $q \leq n$ and the submatrix $\bar{\Upsilon}$ formed from the first $m(q+1)$ rows and the first $r(q+1)$ columns of Υ is a constant matrix.

The proof of part (1) follows from Theorem 4.3, since if Υ is constant then F, F^*, F_1 and F_2 are constant matrices and the system $(\bar{A}, \bar{B}, \bar{C})$ defined by (4.29) is a fixed equivalent of of (A, B, C).

Part (2) of Theorem 4.6 follows similarly from Theorem 4.5.

Example 4.2: Consider the time-variable system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1+3\sin t\cos t & 3\cos^2 t \\ -3\sin^2 t & 1-3\sin t\cos t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} u$$

$$y = [\cos t \quad -\sin t] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For this system

$$Q_c = \begin{bmatrix} \cos t & -(\sin t + \cos t) \\ -\sin t & \sin t - \cos t \end{bmatrix},$$

and

$$(Q_0^+)' = \begin{bmatrix} \cos t & -\sin t \\ \cos t + \sin t & 2\cos t - \sin t \\ 3\cos t + 4\sin t & 4\cos t - 3\sin t \end{bmatrix}.$$

Their product,

$$(Q_0^+)'Q_c = \begin{bmatrix} 1 & -1 \\ 1 & -3 \\ 3 & -7 \end{bmatrix},$$

has rank 2 .

In this case (and in general for single-input-single-output systems)

$$F = Q_0'Q_c = \begin{bmatrix} 1 & -1 \\ 1 & -3 \end{bmatrix}$$

and

$$F^* = (Q_0^*)'Q_c = \begin{bmatrix} 1 & -3 \\ 3 & -7 \end{bmatrix}.$$

Also,

$$F_1 = [1 \quad -1] = [1 \quad 0]\, F$$

and

$$F_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Therefore, $(\bar{A}, \bar{B}, \bar{C})$ is a fixed equivalent of the above system, where

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\bar{C} = [1 \quad 0].$$

4.4 Canonical Forms for Time-Variable Systems

The transformational properties of the controllability and observability matrices will now be utilized to determine the conditions

under which a time-variable system has equivalent realizations in several canonical forms. Methods of transforming to canonical equivalents will also be given when they exist. Only single-input-single-output systems will be considered here, and to avoid the chore of counting the number of derivatives required in various arguments, it will be assumed that all system coefficients are infinitely differentiable.

The matrices of four particular canonical structures are given below:

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \hat{a}_1 & \hat{a}_2 & \hat{a}_3 & \cdots & \hat{a}_n \end{bmatrix}, \quad \hat{b} = \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \\ \vdots \\ \hat{b}_n \end{bmatrix}$$

$$\hat{c} = [1 \quad 0 \quad 0 \quad \cdots \quad 0] \tag{4.32}$$

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \bar{a}_1 & \bar{a}_2 & \bar{a}_3 & \cdots & \bar{a}_n \end{bmatrix} \quad \bar{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\bar{c} = [\bar{c}_1 \quad \bar{c}_2 \quad \bar{c}_3 \quad \cdots \quad \bar{c}_n] \tag{4.33}$$

$$\tilde{A} = \begin{bmatrix} 0 & 0 & \dots & 0 & \tilde{a}_1 \\ 1 & 0 & \dots & 0 & \tilde{a}_2 \\ 0 & 1 & \dots & 0 & \tilde{a}_3 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \tilde{a}_n \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\tilde{c} = [\tilde{c}_1 \quad \tilde{c}_2 \quad \tilde{c}_3 \dots \tilde{c}_n] \tag{4.34}$$

$$\check{A} = \begin{bmatrix} 0 & 0 & \dots & 0 & \check{a}_1 \\ 1 & 0 & \dots & 0 & \check{a}_2 \\ 0 & 1 & \dots & 0 & \check{a}_3 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \check{a}_n \end{bmatrix}, \quad \check{b} = \begin{bmatrix} \check{b}_1 \\ \check{b}_2 \\ \check{b}_3 \\ \vdots \\ \check{b}_n \end{bmatrix}$$

$$\check{c} = [\,0 \quad 0 \quad 0 \dots \quad 1\,] \tag{4.35}$$

All of the above structures have the advantages of requiring at most $2n$ time-variable elements for their realization. Since a general single-input-single-output system of the form (2.1) may have as many as $2n + n^2$ such elements, a considerable saving of hardware may result if a system can be realized or simulated on an analogue computer in one of the canonical forms. It is well known [1] that any fixed completely controllable system has equivalent representations in

$$\tilde{A} = \begin{bmatrix} 0 & 0 & \ldots & 0 & \tilde{a}_1 \\ 1 & 0 & \ldots & 0 & \tilde{a}_2 \\ 0 & 1 & \ldots & 0 & \tilde{a}_3 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & \tilde{a}_n \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\tilde{c} = [\tilde{c}_1 \quad \tilde{c}_2 \quad \tilde{c}_3 \ldots \tilde{c}_n] \tag{4.34}$$

$$\breve{A} = \begin{bmatrix} 0 & 0 & \ldots & 0 & \breve{a}_1 \\ 1 & 0 & \ldots & 0 & \breve{a}_2 \\ 0 & 1 & \ldots & 0 & \breve{a}_3 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & \breve{a}_n \end{bmatrix}, \quad \breve{b} = \begin{bmatrix} \breve{b}_1 \\ \breve{b}_2 \\ \breve{b}_3 \\ \vdots \\ \breve{b}_n \end{bmatrix}$$

$$\breve{c} = [\,0 \quad 0 \quad 0 \ldots \quad 1] \tag{4.35}$$

All of the above structures have the advantages of requiring at most $2n$ time-variable elements for their realization. Since a general single-input-single-output system of the form (2.1) may have as many as $2n + n^2$ such elements, a considerable saving of hardware may result if a system can be realized or simulated on an analogue computer in one of the canonical forms. It is well known [1] that any fixed completely controllable system has equivalent representations in

forms (4. 33) and (4. 34) and that any fixed completely observable system has equivalent representations in forms (4. 32) and (4. 35). As will be shown below, however, this is not the case for time-variable systems. Only uniformly controllable or observable systems possess canonical equivalents.

Forms (4. 32) and (4. 33) are of particular interest. System (4. 32) is the standard state-variable representation of an input-output differential equation [3] , and (4. 33) is the "phase-variable" canonical form utilized in many control system problems [11-14]. The former system will also be seen to be valuable in synthesizing prescribed impulse responses.

In the following two sections, the necessary and sufficient conditions for a system to be realizable in forms (4. 32) and (4. 33) will be derived and methods of constructing the transformations of coordinates required will be given. As forms (4. 34) and (4. 35) are dual to (4. 32) and (4. 33) respectively, it is not necessary to treat them separately.

4. 5 The Input-Output Differential Equation

In this section, the conditions under which system (2. 1) has an equivalent representation of the form (4. 32) will be derived. It will first be shown, however, that this problem is the same as finding the conditions under which (2. 1) is equivalent to a differential equation of the form

$$\sum_{i=1}^{n+1} a_i y^{(i-1)} = \sum_{i=1}^{n} b_i u^{(i-1)} \tag{4.36}$$

where $a_{n+1} = 1$, and the coefficients a_i and b_i for $i = 1, 2, \ldots, n$ are continuous functions.

The equivalence of (4.32) and (4.36) is well known [3] and their coefficients are related by the equations

$$a_i = -\hat{a}_i , \qquad i = 1, 2, \ldots, n$$

and

$$b_i = \hat{b}_{n-i+1} + \sum_{j=1}^{n-i} \sum_{h=0}^{n-i-j+1} \binom{h+i-1}{i-1} a_{i-j-h} \frac{d^h}{dt^h} \hat{b}_j ,$$

$$i = 1, 2, \ldots, n-1$$

$$b_n = \hat{b}_1 .$$

The converse problem, that of expressing a system in input-output differential equation form given its state-variable representation has been considered rather intractable by several authors [15, 16]. Recently, Weiss and Kalman [17] have presented a solution which utilizes the impulse response of the given system. The conditions they derive for existence of the input-output differential equation are somewhat imprecise, however. We will solve this problem below by transforming the system to form (3.32) which in light of the above discussion,

is all that is required. Most importantly, this solution does not require knowledge of the system impulse response.

The following theorem provides a necessary and sufficient condition for a system to have a canonical equivalent of the form (4.32).

Theorem 4.7: System (2.1) (with $m = r = 1$) is equivalent to a system of the form (4.32) if and only if (2.1) is uniformly observable.

Let Q_0 be the observability matrix of (2.1) and $\hat{Q}_0$, the observability matrix of (4.32). By inspection $\hat{Q}_0 = I_n$, which permits the following compact proof.

Suppose, (2.1) is equivalent to (4.32) then from (4.15) a nonsingular continuously differentiable matrix T exists such that

$$Q_0 = T'\hat{Q}_0 = T'$$

Therefore, (2.1) must be uniformly observable.

Conversely, if (2.1) is uniformly observable then $T = Q_0'$ qualifies as an equivalence transformation. Under this transformation of coordinates,

$$\begin{aligned}\hat{A}' &= (-\frac{d}{dt}\hat{Q}_0 + \hat{Q}_0^*)\hat{Q}_0^{-1} \\ &= Q_0^* = Q_0^{-1}Q_0^* .\end{aligned}$$

Therefore,

$$\hat{A}' = \begin{bmatrix} 0 & 0 & \cdots & 0 & \hat{a}_1 \\ 1 & 0 & \cdots & 0 & \hat{a}_2 \\ 0 & 1 & \cdots & 0 & \hat{a}_3 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \hat{a}_n \end{bmatrix}$$

where

$$\hat{a} = \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \vdots \\ \hat{a}_n \end{bmatrix} = Q_0^{-1} s_n .$$

Also,

$$\hat{c} = c(Q_0')^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

which completes the proof.

It should be observed that the above proof also provides an explicit method for finding form (4. 32) when it exists, which is summarized below:

(1) $z = Q_0'x$

(2) $\hat{a} = Q_0^{-1} s_n$

(3) $\hat{b} = Q_0' b$.

Note that contrary to the results of [17], total observability is not sufficient for the existence of the input-output differential equation. For example, the system

$$\dot{x} = u$$

$$y = (\sin t)x$$

cannot be transformed to form (4.36).

To demonstrate the value of transforming to canonical form for analogue computer simulation or analysis we consider the following system:

<u>Example 4.3:</u>

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \cos t \sin t & 1 + \cos^2 t & \sin t \\ -(1 + \sin^2 t & -\cos t \sin t & \cos t \\ -(\cos t + 2\sin t) & \sin t - 2\cos t & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \cos t + \sin t \\ \cos t - \sin t \\ -6 \end{bmatrix} u$$

$$y = \begin{bmatrix} \cos t & -\sin t & 0 \end{bmatrix} x$$

The observability matrix of this example is

$$Q_0 = \begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and since it has rank 3 for all t, transformation to form (4.32) is possible. If $z = Q_0' x$, it can be verified that

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} z .$$

Thus by transforming to canonical form, we obtain a fixed equivalent in this case. It can be shown in general via Theorem 4.4, that if a uniformly controllable and observable system possesses a fixed equivalent, its canonical representations will be fixed. This provides further motivation for transforming to these system structures when possible.

4.6 The Phase-Variable Canonical Form

The problem of transforming system (2.1) to an equivalent phase-variable canonical form is a considerably more difficult problem

than that considered in the previous section. The reason for this is that the controllability matrix of (4. 33) is a function of the coefficients $\bar{a}_i$. In the fixed case, Kalman [1] has shown that a necessary and sufficient condition for such an equivalent to exist is that system (2. 1) be completely controllable. It will be shown here that under somewhat more restrictive conditions (uniform controllability) a system of the form (4. 33) equivalent to system (2. 1) can also be found in the time-variable case. In addition, a general method is given for constructing both the equivalent canonical system and the transforming matrix T. The specialization of this method to fixed systems is of independent interest as it proves to be considerably simpler than previously published techniques [4], [5], [6]. Johnson and Wonham [4] considered the problem of finding T when (2. 1) is fixed and A has distinct eigenvalues, and Mufti [5] extended their technique to the case of multiple eigenvalues. Further refinement of the methods of [4] and [5] was made by Chidambara [6]. The methods of constructing T employed in [4], [5] and [6], however, involve finding the eigenvalues and eigenvectors of A, a computational task equivalent to solving the differential equations represented by system (2. 1). It will be seen that the method developed here is independent of the eigenvalues of A and that it provides simple explicit forms for the matrices $\bar{a}$, T and T^{-1}.

Let Q, and $\bar{Q}$ be the controllability matrices of (2. 1) and (4. 33) respectively.

Matrices $\bar{Q}$ and $\bar{p}_n$ will be examined closely, for it will be shown that they serve to determine the transformation from (2. 1) to (4. 33) when it exists. It can be verified by direct construction that

$$\bar{Q} = \begin{bmatrix} 0 & 0 & \cdots & 0 & (-1)^{n-1} \\ 0 & 0 & \cdots & (-1)^{n-2} & q_{n-1,n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & -1 & \cdots & q_{n-2,2} & q_{n-1,2} \\ 1 & q_{11} & \cdots & q_{n-2,1} & q_{n-1,1} \end{bmatrix}, \quad \bar{p}_n = \begin{bmatrix} q_{n,n} \\ q_{n,n-1} \\ \vdots \\ q_{n,2} \\ q_{n,1} \end{bmatrix}, \tag{4. 37}$$

where

$$\begin{aligned} q_{ik} &= -q_{i-1,k-1} + \dot{q}_{i-1,k} & 1<k<i\leq n \\ &= (-1)^i \bar{a}_{n-i+1} - \sum_{j=0}^{i-2} \bar{a}_{n-j} q_{i-1,j+1} + \dot{q}_{i-1,1}\,, & k = 1<i\leq n \\ &= (-1)^i \bar{a}_n\,, & 1\leq k = i\leq n\,. \end{aligned} \tag{4. 38}$$

From the form of $\bar{Q}$ it is clear that any system of the form (4. 33) is uniformly controllable. A more informative relation for q_{ik} can be easily derived from (4. 38) as

$$q_{ik} = (-1)^i \bar{a}_{n-i+k} + (-1)^k \sum_{j=0}^{i-k+1} \bar{a}_{n-j} q_{i-k,j+1}$$

$$+ \sum_{j=1}^{k} (-1)^{j+1} \dot{q}_{i-j,k-j+1} \;, \quad 1<k<i\leq n \qquad (4.39)$$

and

$$q_{ii} = (-1)^i \bar{a}_n \,, \quad 1\leq i\leq n \,.$$

It follows by a simple induction argument that

$$q_{ik} = (-1)^i \bar{a}_{n-i+k} + \left\{ \begin{matrix} \text{terms involving only the} \\ \text{coefficients } \bar{a}_n, \ldots, \bar{a}_{n-i+k+1} \end{matrix} \right\}, \quad 1\leq i\leq k\leq n \qquad (4.40)$$

For notational convenience, the bracketed expression in (4.40) will be represented by the symbol θ_{i-k}. That is, any function that can be expressed solely in terms of the coefficients $\bar{a}_n, \ldots, \bar{a}_{n-r+1}$ will be replaced by the symbol θ_r wherever no other information about the function is needed. With this notation (4.39) becomes

$$q_{ik} = (-1)^i \bar{a}_{n-i+k} + \theta_{i-k} = \theta_{i-k+1} \,, \quad 1\leq i\leq k\leq n \,, \quad \theta_0 = 0 \,. \qquad (4.41)$$

It is now possible to prove

<u>Theorem 4.8:</u> System (2.1) is equivalent to a system of the form (4.33) if and only if (2.1) is uniformly controllable.

The necessity of the controllability condition is easily established, since if (2.1) is equivalent to (4.33) where $z = Tx$, then

$$\bar{Q} = TQ \tag{4.42}$$

But $\bar{Q}$ and T have rank n everywhere, therefore Q must have rank n everywhere, which implies that system (2.1) is uniformly controllable. Conversely, if (2.1) is uniformly controllable the matrix

$$T = \bar{Q}\,Q^{-1} \tag{4.43}$$

is nonsingular when $\bar{Q}$ is the controllability matrix of any system of the form (4.33). Moreover Property 4.1 shows that (4.43) must be the form of the transforming matrix if it exists. Thus to prove that the uniform controllability condition is sufficient let $z = Tx$ where T is given by (4.43). If the relationships

$$\bar{A} = (TA + \dot{T})T^{-1} \tag{4.44}$$

and

$$\bar{b} = Tb \tag{4.45}$$

can be satisfied for some unique set of coefficients $\bar{a}_i$ the desired equivalence will then be established. Note that (4.45) is valid by inspection, and (4.44) can be rewritten as

$$\bar{A} = (\bar{Q}\tilde{A} + \frac{d}{dt}\bar{Q})\bar{Q}^{-1} \tag{4.46}$$

in which

$$\tilde{A} = (Q^{-1}A + \frac{d}{dt}Q^{-1})Q = -Q^{-1}(-AQ + \frac{d}{dt}Q) = -Q^{-1}Q^{*}$$

$$= \begin{bmatrix} 0 & 0 & \dots & 0 & g_1 \\ -1 & 0 & \dots & 0 & g_2 \\ 0 & -1 & \dots & 0 & g_3 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & -1 & g_n \end{bmatrix}$$

and $g = -Q^{-1}p_n$, where g_i is the i-th element of the vector g.*

From (4.46) it is clear that

$$-\bar{A}\bar{Q} + \frac{d}{dt}\bar{Q} = -\bar{Q}\tilde{A}$$

or

$$[\bar{p}_1\bar{p}_2 \dots \bar{p}_{n-1}\bar{p}_n] = [\bar{p}_1\bar{p}_2 \dots \bar{p}_{n-1}(-\bar{Q}g)] .$$

Therefore, it will suffice to satisfy

---

*Note that $g = -\tilde{a}$, as defined by (4.34).

$$\bar{p}_n = -\bar{Q}\, g \quad . \tag{4.47}$$

It follows from the representation of the elements of $\bar{Q}$ and $\bar{p}_n$ given in (4. 37) that the first row of (4. 47) is $\bar{a}_n = g_n$ and that remaining rows can be written in the (symbolic) form

$$(-1)^{n}\bar{a}_r + \theta_{n-r} = (-1)^r g_r + g_{r+1}\theta_1 + \ldots + g_n\theta_{n-4} \, ,$$

$$\geq 1 \; . \tag{4.48}$$

Since $\theta_1, \ldots, \theta_{n-r}$ represent terms involving only the coefficients $\bar{a}_n \ldots, \bar{a}_{r+1}$, equation (4. 48) determines $\bar{a}_r$ uniquely as a function of the known $g_r, \ldots, g_n$ and previously calculated $\bar{a}_i$ for $i > r$. That is, the $\bar{a}_i$ can be found recursively beginning with $\bar{a}_n = g_n$. The $\bar{a}_i$ will also be infinitely differentiable since the operations involved in (4. 48) are multiplication, addition and differentiation of the functions g_i which are infinitely differentiable due to the assumptions on system (2. 1). The matrix T found by substituting the recursively determined $\bar{a}_i$ into (4. 43) is therefore nonsingular and continuously differentiable, and system (2. 1) is equivalent to a system of the form (4. 33).

It should be emphasized that the proof above provides a direct method for finding the canonical system and the transforming matrix. To illustrate the method, the following third order system will be considered.

Example 4.4: Let the coefficient matrices of system (2.1) be

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & -1 \\ -e^{-t} & e^{-t} & -2 \end{bmatrix}, \quad b = \begin{bmatrix} e^{-t} \\ 0 \\ 0 \end{bmatrix}$$

The matrices Q, p_3, and g are then

$$Q = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & 0 & e^{-2t} \\ 0 & e^{-2t} & 0 \end{bmatrix}, \quad p_3 = -\begin{bmatrix} e^{-t} \\ e^{-2t} \\ e^{-3t} \end{bmatrix}, \quad g = \begin{bmatrix} e^{-t} \\ e^{-t} \\ 1 \end{bmatrix}.$$

It can easily be established from (4.48) that for $n = 3$

$$\bar{a}_3 = g_3, \quad \bar{a}_2 = -g_2 + 2\dot{g}_3, \quad \bar{a}_1 = g_1 - \dot{g}_2 + \ddot{g}_3$$

so for this example,

$$\bar{a} = \begin{bmatrix} 2e^{-t} \\ -e^{-t} \\ 1 \end{bmatrix} \text{ and } \bar{Q} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & -1 & 1-e^{-t} \end{bmatrix}.$$

The matrices T and $\bar{A}$ are thus given by

$$T = \begin{bmatrix} 0 & e^{2t} & 0 \\ 0 & e^{2t} & -e^{2t} \\ e^{t} & e^{2t}-e^{t} & -e^{2t} \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2e^{-t} & -e^{-t} & 1 \end{bmatrix}.$$

For fixed systems the transformation method outlined above greatly simplifies, and explicit expressions may be obtained for both the coefficients of the canonical system and the transforming matrix. The recurrence relation between the columns of the controllability matrix may be given in the form

$$p_{k+1} = Ap_k , \quad p_0 = b , \tag{4.49}$$

and from the fixed equivalent of (4.47) it is found that

$$g = Q^{-1} p_n = Q^{-1} A^n b \tag{4.50}$$

For fixed systems, the elements of both g and $\bar{a}$ are the coefficients of the characteristic equation of A^* so that

$$\bar{a} = g = Q^{-1} A^n b , \tag{4.51}$$

completely specifying the canonical system (4.33). Once the characteristic equation is found, the transformation T is given

---

*As are $\check{a}$, $\hat{a}$, and $\tilde{a}$.

directly and explicitly by (4.43), where

$$\hat{Q} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & c_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & c_{n-3} & c_{n-2} \\ 1 & c_1 & \cdots & c_{n-2} & c_{n-1} \end{bmatrix}, \qquad (4.52)$$

and

$$c_k = \sum_{i=0}^{k-1} \bar{a}_{n-i} c_{k-i-1}, \qquad k = 1, 2, \ldots, n; \quad c_0 = 1 .$$

An even simpler form may be given for the inverse of T ($K = T^{-1}$ is the form of the transformation computed in references [4]-[6]) since

$$\hat{Q}^{-1} = \begin{bmatrix} -\bar{a}_2 & -\bar{a}_3 & \cdots & -\bar{a}_n & 1 \\ -\bar{a}_3 & -\bar{a}_4 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ -\bar{a}_n & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

as may be easily verified. It is clear then that

$$K = T^{-1} = Q\hat{Q}^{-1} \tag{4.53}$$

requires no further calculation other than the multiplication of Q and $\overline{Q}^{-1}$ once the characteristic equation of A is known.

In order to compare the computational value of the above procedure with existing techniques the following example treated by Mufti [5] and Chidambara [6] will be considered.

Example 4.5: Let the coefficient matrices of system (2.1) be

$$A = \begin{bmatrix} 1 & 6 & -3 \\ -1 & -1 & 1 \\ -2 & 2 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

From (4.37)

$$Q = \begin{bmatrix} 1 & 4 & -2 \\ 1 & -4 & -3 \\ 1 & 0 & -10 \end{bmatrix}, \quad p_3 = \begin{bmatrix} 10 \\ -5 \\ -2 \end{bmatrix}$$

The coefficients of the characteristic equation are found with (4.51) as

$$\overline{a} = \frac{1}{36} \begin{bmatrix} 10 & 40 & -14 \\ 7 & -8 & 1 \\ 1 & 4 & -5 \end{bmatrix} \begin{bmatrix} 10 \\ -5 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$$

Thus

$$\overline{Q}^{-1} = \begin{bmatrix} -3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

so that from (4. 53),

$$K = T^{-1} = \begin{bmatrix} 1 & 4 & -2 \\ 1 & -4 & -3 \\ 1 & 0 & -10 \end{bmatrix} \begin{bmatrix} -3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -5 & 4 & 1 \\ -6 & -4 & 1 \\ -13 & 0 & 1 \end{bmatrix}.$$

References

[1] R. E. Kalman, "Mathematical Description of Linear Dynamical Systems," J. SIAM Control, vol. 1, pp. 152-192; 1963.

[2] R. J. Schwarz and B. Friedland, Linear Systems, McGraw-Hill Book Co., Inc., New York, N. Y.; 1965.

[3] L. A. Zadeh and C. A. Desoer, Linear System Theory, McGraw-Hill Book Co., Inc., New York, N. Y.; 1963.

[4] C. D. Johnson and W. M. Wonham, "A Note on the Transformation to Canonical (Phase-Variable) Form," IEEE Trans. on Automatic Control, vol. AC-9, pp. 312-313; July, 1964.

[5] I. H. Mufti, "On the Reduction of a System to Canonical (Phase-Variable) Form," IEEE Trans. on Automatic Control, vol. AC-10, pp. 206-207; April, 1965.

[6] M. R. Chidambara, "The Transformation to (Phase-Variable) Canonical Form, "IEEE Trans. on Automatic Control, vol. AC-10 pp. 492-494; October, 1965.

[7] A. C. Aitken, Determinants and Matrices, Oliver and and Boyd Ltd., London; 1939.

[8] H. J. Wertz, "On the Numerical of a Recurrent Problem: The Vandermonde Matrix," IEEE Trans. on Automatic Control, vol. AC-9, p. 492; October, 1965.

[9] D. C. Youla, "The Synthesis of Linear Dynamical Systems from Prescribed Weighting Patterns," PIB Report MRI-1271-65, Department of Electrophysics Polytechnic Institute of Brooklyn, June 1965.

[10] A. R. Stubberud, "Reducibility of Linear Systems," IEEE Trans. on Appl. and Ind., vol. 67, pp. 214-219; July, 1963.

[11] R. E. Kalman, "Lyapunov Functions for the Problem of Lurie in Automatic Controls," Proc. Nat. Acad. Sci. USA, vol. 49, pp. 201-205; 1963.

[12] B. S. Morgan, "The Synthesis of Single Variable Systems by State Variable Feedback," Proc. Allerton Conf. on Circuit and System Theory, University of Illinois, Urbana; November, 1963.

[13] C. D. Johnson and J. E. Gibson, "Optimal Control with Quadratic Performance Index and Fixed Terminal Time," IEEE Trans. on Automatic Control, vol. AC-9, pp. 355-360; October, 1964.

[14] R. W. Bass and I. Gura, "High Order System Design Via State-Space Considereations," Preprints JACC, Troy, N.Y.; June, 1965.

[15] B. K. Kinariwala, "Analysis of Time-Varying Networks," IRE Conv. Record, vol. 9, part 4, 1961, pp. 268-276.

[16] R. K. Keenan, "Free Oscillations in Periodically Varying Systems," Report no. MEE 65-2, School of Engineering, Monash University; Australia; April, 1965.

[17] L. Weiss and R. E. Kalman, "Contributions to Linear System Theory," Technical Report 64-9, Research Institute for Advanced Study, Maryland; March, 1964.

[18] L. M. Silverman and H. E. Meadows, "Degrees of Controllability in Time-Variable Linear Systems," Proceedings of the National Electronics Conf., vol. 21, pp. 689-693; 1965.

[19] L. M. Silverman, "Transformation of Time-Variable Systems to Canonical (Phase-Variable) Form," IEEE Trans. on Automatic Control, to appear.

CHAPTER 5

REALIZATION OF IMPULSE RESPONSE MATRICES

5.1 Introduction

In this chapter we will be concerned with the realization (or synthesis) of prescribed impulse response matrices as systems of the form (2.1). Various aspects of this problem have been treated by Gilbert [1], Kalman [2, 3], Weiss and Kalman [4], Youla [5], and Desoer [6], and earlier for the scalar case by Darlington [7, 8], Batkov [9] and Borskii [10]. We shall present what is believed to be a new approach based on the properties of equivalent systems derived in the previous chapter and on a method of Borskii [10] for synthesizing scalar impulse response functions.

Our main emphasis will be on the realization of non-stationary response matrices $H(t, \tau)$ and in contrast to previous work, we will not assume a known separated form for $H(t, \tau)$ (i.e., $H(t, \tau) = \Psi(t)\Theta(\tau)$, where $\Psi(t)$ and $\Theta(t)$ are finite matrices). The method to be presented here, starts with a non-separated impulse matrix and yields a uniformly controllable and uniformly observable system, when such exists.

The specialization of this method to the realization of stationary impulse response matrices is of independent interest. It leads to an easily tested necessary and sufficient condition for a stationary matrix to be the impulse response matrix of a minimal (completely

controllable and completely observable) fixed system of specified order, together with a systematic procedure for constructing minimal realizations.

5.2 Realization of Time-Variable Impulse Response Matrices

Let $H(t,\tau)$ be an $m \times r$ continuous matrix function of two variables. It will be said that $H(t,\tau)$ is realizable if there exists a system (A, B, C) for which $H(t,\tau)$ is the impulse response matrix. Any system having $H(t,\tau)$ as its impulse response matrix is said to realize $H(t,\tau)$.

It has been shown by Kalman [2] that a necessary and sufficient condition for $H(t,\tau)$ to be realizable is that there exist continuous matrices of finite dimension, $\Psi(t)$ and $\Theta(t)$, such that

$$H(t,\tau) = \begin{cases} \Psi(t)\Theta(\tau), & t \geq \tau \\ 0, & t < \tau \end{cases} \tag{5.1}$$

The proof of this condition is quite straightforward. It is clear that the system $(0, \Theta, \Psi)$ will realize (5.1), and the impulse response matrix of any system of the form (A, B, C) is given as

$$H(t,\tau) = \begin{cases} C(t)X(t)X^{-1}(\tau)B(\tau), & t \geq \tau \\ 0 & t < \tau \end{cases}$$

where $X(t)$ is a fundamental matrix for the system.

If $H(t,\tau)$ satisfies (5.1), it will be said to be separable. The major problem we will consider in this chapter is the realization of response matrices which are not given in separable form, or even known to be separable a priori. There are several known procedures

for realizing fixed systems from non-separated response matrices (usually from the transfer function matrix - an equivalent specification to a non-separated impulse response matrix) [1, 2, 3, 6], but the general time-variable case does not seem to have been treated previously. We will present a procedure which starts with an arbitrary matrix function of two-variables and yields when such exists, a uniformly observable and uniformly controllable realization. The specialization of this synthesis procedure to fixed systems is of independent interest, as it has several advantages compared with previously published techniques.

Assuming that $H(t,\tau)$ is infinitely differentiable in both of its arguments, the following matrices may be defined for all $i, j \geq 0$ and $t \geq \tau$:

$$H_{ij}(t,\tau) = \frac{\partial^i}{\partial t^i}\frac{\partial^j}{\partial \tau^j} H(t,\tau)$$

$$\Gamma_{ij}(t,\tau) = \begin{bmatrix} H_{00}(t,\tau) & H_{01}(t,\tau) & \cdots & H_{0,j-1}(t,\tau) \\ H_{10}(t,\tau) & H_{11}(t,\tau) & \cdots & H_{1,j-1}(t,\tau) \\ \vdots & \vdots & & \vdots \\ H_{i-1,0}(t,\tau) & H_{i-1,1}(t,\tau) & \cdots & H_{i-1,j-1}(t,\tau) \end{bmatrix}$$

If $H(t,\tau)$ is realizable by an n^{th} order system (A, B, C) then it is easily verified that

$$H_{ij}(t,\tau) = S_i'(t)\Phi(t,\tau)P_j(\tau) \qquad t \geq \tau$$

where $\Phi(t,\tau)$ is the transition matrix of (A, B, C) and S_i and P_j are as defined in Chapter 3. Thus, if W_0 and W_c are the Wronskian matrices of (A, B, C) then

$$\Gamma_{nn}(t,\tau) = W_0(t)W_c(\tau) \qquad t \geq \tau$$

and

$$\Gamma_{nn}(t,t) = Q_0'(t)Q_c(t) = \Gamma(t,t) \tag{5.2}$$

Similarly,

$$\Gamma_{n+1,n+1}(t,t) = (Q_0^+(t))'Q_c^+(t) = \Upsilon(t)\,. \tag{5.3}$$

The importance of Theorem 4.5 to the realization problem is now clear. Suppose we know that $H(t,\tau)$ is realizable by a system of order $\bar{n}$ (not necessarily a least order realization). The Υ matrix of an $\bar{n}^{th}$-order realization is then directly obtainable with (5.3) so that Theorem 4.5 can be utilized to construct a uniformly controllable and observable realization if such exists.

While in many cases it is possible to determine $\bar{n}$ by inspection (this is true for all stationary impulse response matrices

[2]) it is desirable to have a systematic realization procedure which includes the determination of the least $n < \bar{n}$. Observe that if $H(t, \tau)$ is realizable by a member of U_n, there exist integers $\ell \leq n$ and $k \leq n$ such that if $i \geq \ell$ and $j \geq k$ then $\Gamma_{ij}(t, \tau)$ has rank n for all $t \geq \tau$. The integers ℓ and k are the least such that

$$[S_0 \vdots S_1 \vdots \ldots \vdots S_{\ell-1}]$$

and

$$[P_0 \vdots P_1 \vdots \ldots P_{k-1}]$$

have rank n for all t. It will be shown below that under certain conditions the converse is also true. That is, by examining the ranks of the sequence of matrices $\Gamma_{ij}(t, \tau)$, it is possible to determine whether $H(t, \tau)$ is realizable as a member of U_n. Furthermore, an explicit separation and realization procedure will be given.

To simplify the presentation, we will first consider the case where $H(t, \tau) = h(t, \tau)$, a scalar function of two variables.

<u>Theorem 5.1:</u> If for some finite, non-negative integer n the matrices $\Gamma_{nn}(t, \tau)$ and $\Gamma_{n+1, n+1}(t, \tau)$ have rank n for all t and τ, $t \geq \tau$, then $h(t, \tau)$ is realizable as an n^{th} order uniformly controllable and uniformly observable system.

The proof of this theorem together with a canonical realization of $h(t,\tau)$ is given below.

If $\Gamma_{nn}(t,\tau)$ and $\Gamma_{n+1,n+1}(t,\tau)$ have rank n for all $t \geq \tau$, then certainly the last row of $\Gamma_{n+1,n+1}(t,\tau)$ must be a linear combination of the first n rows for all t and τ. That is,

$$h_{nj}(t,\tau) = \sum_{i=0}^{n-1} a_{i+1}(t,\tau)h(t,\tau)\,; \qquad j = 0, 1, \ldots, n$$

for some set of coefficients $a_i(t,\tau)$, $i = 1, 2, \ldots, n$. As will now be shown, the $a_i(t,\tau)$ are unique, continuously differentiable functions of t and do not depend on τ. Let

$$F(t,\tau) = \Gamma_{nn}(t,\tau) \quad , \tag{5.4}$$

$$F^*(t,\tau) = \frac{\partial}{\partial t} F(t,\tau), \tag{5.5}$$

and

$$A(t,\tau) = \begin{bmatrix} 0 & 1 & 0 & \ldots 0 \\ 0 & 0 & 1 & \ldots 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 \\ a_1(t,\tau) & a_2(t,\tau) & a_3(t,\tau) & \ldots a_n(t,\tau) \end{bmatrix} . \tag{5.6}$$

Since $F^*(t,\tau)$ has as its rows the last n rows of $\Gamma_{n+1,n}(t,\tau)$, it is clear that

$$F^*(t,\tau) = \frac{\partial}{\partial t} F(t,\tau) = A(t,\tau)\, F(t,\tau) \tag{5.7}$$

Furthermore, $F(t,\tau)$ has rank n for all $t \geq \tau$ so that equation (5.7) possesses the unique continuously differentiable solution

$$A(t,\tau) = F^*(t,\tau) F^{-1}(t,\tau). \tag{5.8}$$

To establish that $A(t,\tau)$ is not a function of τ, differentiate both sides of (5.8) with respect to τ:

$$\frac{\partial}{\partial \tau} A(t,\tau) = \left[\frac{\partial}{\partial \tau} F^*(t,\tau)\right] F^{-1}(t,\tau) - F^*(t,\tau) F^{-1}(t,\tau)\left[\frac{\partial}{\partial \tau} F(t,\tau)\right] F^{-1}(t,\tau). \tag{5.9}$$

Since $\frac{\partial}{\partial \tau} F(t,\tau)$ and $\frac{\partial}{\partial \tau} F^*(t,\tau)$ are formed from the last n rows and columns of $\Gamma_{n,n+1}(t,\tau)$ and $\Gamma_{n+1,n+1}(t,\tau)$, respectively, it must also be true that

$$\frac{\partial}{\partial \tau} F^*(t,\tau) = A(t,\tau) \frac{\partial}{\partial \tau} F(t,\tau). \tag{5.10}$$

Therefore,

$$\frac{\partial}{\partial\tau} A(t,\tau) = A(t,\tau)[\frac{\partial}{\partial\tau} F(t,\tau)] F^{-1}(t,\tau)$$

$$-A(t,\tau) F(t,\tau) F^{-1}(t,\tau)[\frac{\partial}{\partial\tau} F(t,\tau)] F^{-1}(t,\tau)$$

$$= 0 .$$

Thus, with a slight abuse of notation,

$$A(t,\tau) = A(t) = F^{*}(t,\tau) F^{-1}(t,\tau) \tag{5.11}$$

for any $\tau \le t$. For convenience, let $\tau = t$ in (5.11) so that the defining equation for $A(t)$ is

$$A(t) = F^{*}(t,t) F^{-1}(t,t) . \tag{5.12}$$

It will now be shown that $F(t,\tau)$ can be separated in the form

$$F(t,\tau) = F(t,\lambda) F^{-1}(\tau,\lambda) F(\tau,\tau) . \tag{5.13}$$

for any arbitrary parameter λ.

Observe first that $F^{-1}(t,\lambda) F(t,\tau)$ is not a function of t, since

$$\frac{\partial}{\partial t}[F^{-1}(t,\lambda)F(t,\tau)] = -F^{-1}(t,\tau)[\frac{\partial}{\partial t}F(t,\lambda)]F^{-1}(t,\lambda)F(t,\tau)$$

$$+ F(t,\lambda)\frac{\partial}{\partial t}F(t,\tau)$$

$$= -F(t,\lambda)A(t)F(t,\lambda)F^{-1}(t,\lambda)F(t,\tau)$$

$$+ F^{-1}(t,\lambda)A(t)F(t,\tau)$$

$$= 0 \ .$$

In particular, therefore,

$$F^{-1}(t,\lambda)F(t,\tau) = F^{-1}(\tau,\lambda)F(\tau,\tau) \ ,$$

which establishes (5.13).

Thus, $h(t,\tau)$ is separable in the form

$$h(t,\tau) = [e_1' F(t,\lambda)][F^{-1}(\tau,\lambda)F(\tau,\tau)e_1]$$

for any λ where e_1' is the n-dimensional row vector

$$e_1' = [\,1 \quad 0 \ . \ . \ . \ 0] \ .$$

One realization of $h(t,\tau)$ is immediately obtained as

$$\dot{z} = F^{-1}(t,\lambda)F(t,t)e_1\, u(t)$$

$$y = e_1' F(t,\lambda) z \quad . \tag{5.14}$$

This realization is without feedback, however, which for many applications may be undesirable. Furthermore, even if $h(t,\tau)$ is stationary (i.e., $h(t,\tau) = h(t-\tau)$), (5.14) will be a time-variable system.

A more useful realization is obtained by observing that for any λ, $F(t,\lambda)$ is a fundamental matrix for

$$\dot{x} = A(t)x \quad , \tag{5.15}$$

where $A(t)$ is given by (5.12). Therefore, the transition matrix of (5.15) is given by

$$\phi(t,\tau) = F(t,\lambda)F^{-1}(\tau,\lambda). \tag{5.16}$$

The impulse response $h(t,\tau)$ can thus be expressed as

$$h(t,\tau) = e_1'\phi(t,\tau)F(\tau,\tau)e_1 \tag{5.17}$$

and is clearly realized by the system in the canonical form (4. 32)

$$\dot{x} = A(t)x + b(t)u$$

$$y = cx \tag{5.8}$$

where

$$b(t) = F(t,t)e_1$$

$$c = e_1'$$

$$A(t) = F^*(t,t)F^{-1}(t,t) \quad . \tag{5.19}$$

It is evident from (5. 2) that the realizations obtained by this method are uniformly controllable and uniformly observable, since if $Q_c(t)$ and $Q_0(t)$ are the controllability and observability matrices of (5. 18) then

$$Q_0(t) = I_n$$

and

$$Q_c(t) = F(t,t) \quad .$$

An important property of the realization (5.18) is that if $h(t, \tau)$ happens to be stationary the system will of necessity be time-invariant. This is obvious, since if $h(t, \tau) = h(t - \tau)$ then it must also be true that

$$F(t, \tau) = F(t - \tau)$$

and

$$F^*(t, \tau) = F(t - \tau)$$

so that

$$b = F(0) e_1$$

and

$$A = F^*(0) F^{-1}(0) \quad .$$

<u>Example 5.1</u> Suppose we are given the scalar function

$$h(t, \tau) = e^{-t} \sin(t-\tau), \qquad t \geq \tau$$

Observe that

$$\Gamma_{22}(t,\tau) = \begin{bmatrix} e^{-t}\sin(t-\tau) & -e^{-t}\cos(t-\tau) \\ e^{-t}(\cos(t-\tau)-\sin(t-\tau)) & e^{-t}(\cos(t-\tau)+\sin(t-\tau)) \end{bmatrix}$$

and

$$\det \Gamma_{22}(t,\tau) = e^{-2t} . \qquad t \geq \tau$$

Also,

$$\Gamma_{33}(t,\tau) = \begin{bmatrix} e^{-t}\sin(t-\tau) & -e^{-t}\cos(t-\tau) & e^{-t}\sin(t-\tau) \\ e^{-t}(\cos(t-\tau)-\sin(t-\tau)) & e^{-t}(\cos(t-\tau)+\sin(t-\tau)) & e^{-t}(\cos(t-\tau)-\sin(t-\tau)) \\ -2e^{-t}\cos(t-\tau) & -2e^{-t}\sin(t-\tau) & -2e^{-t}\cos(t-\tau) \end{bmatrix}$$

and clearly

$$\det \Gamma_{33}(t,\tau) = 0 ,$$

so that $\Gamma_{33}(t,\tau)$ also has rank 2 for all $t \geq \tau$.

Let $F(t,\tau) = \Gamma_{22}(t,\tau)$; then

$$F(t,t) = \begin{bmatrix} 0 & -e^{-t} \\ e^{-t} & e^{-t} \end{bmatrix}$$

and

$$F^*(t,t) = \begin{bmatrix} e^{-t} & e^{-t} \\ -2e^{-t} & 0 \end{bmatrix}.$$

Applying (5.19), one sees that $h(t,\tau)$ is realized by the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

It should be clear that the realization procedure above is not applicable to all realizable impulse response matrices. If a system is not uniformly controllable and observable, or reducible to such a system, its impulse response matrix will <u>not</u> satisfy the conditions of Theorem 5.1.

Matrices equivalent to $\Gamma_{nn}(t)$ and $\Gamma_{n+1,n+1}(t)$ were first used by Borskii [10] to realize single-input single-output systems. Borskii's methods and proofs are somewhat vague, however, and lead to several false conclusions (such as a statement that the criteria of Theorem 5.1 are also necessary for realizability). Furthermore, Borskii's form of realization is an input-output differential equation,

which results in unnecessary complication in finding the system coefficients.

5.3 Realization of Multi-Input Multi-Output Impulse Response Matrices

We will now consider the general problem of realizing non-separated matric functions of two variables. The major result is summarized in the following theorem.

Theorem 5.2: Suppose that for some finite positive integers ℓ, k and n ($\ell \le n$, $k \le n$), the matrices $\Gamma_{\ell k}(t, \tau)$ and $\Gamma_{\ell+1, k+1}(t, \tau)$ have rank n for all $t \ge \tau$, and an $n \times n$ submatrix of $\Gamma_{\ell k}(t, \tau)$ also has rank n for all $t \ge \tau$. Then, $H(t, \tau)$ is realizable as an n^{th} order uniformly observable and uniformly controllable system.

A constructive proof of this theorem is given below.

If $F(t, \tau)$ is an $(n \times n)$ submatrix of $\Gamma_{\ell k}(t, \tau)$ having rank n for all t and τ then there must exist matrices $C_1(t, \tau)$ and $B_2(t, \tau)$ such that

$$H(t, \tau) = C_1(t, \tau) F(t, \tau) B_1(t, \tau) . \qquad (5.20)$$

Let $F_1(t, \tau)$ be the matrix composed of those columns of the first m rows of $\Gamma_{\ell k}(t, \tau)$ which correspond to the columns of $F(t, \tau)$. Then it is clear that

$$C_1(t, \tau) = F_1(t, \tau) F^{-1}(t, \tau). \qquad (5.21)$$

Similarly, let $F_2(t,\tau)$ be the matrix composed of those rows of the first r columns of $\Gamma_{\ell k}(t,\tau)$ which correspond to the rows of $F(t,\tau)$. Then,

$$B_1(t,\tau) = F^{-1}(t,\tau)F_2(t,\tau). \tag{5.22}$$

Also, let $F^*(t,\tau)$ be those rows and columns of $\Gamma_{\ell+1,k}(t,\tau)$ corresponding to the rows and columns of $\Gamma_{\ell,k}(t,\tau)$ composing $F(t,\tau)$. Then

$$F^*(t,\tau) = \frac{\partial}{\partial t}F(t,\tau) = A(t,\tau)F(t,\tau) .$$

As in the scalar case, $A(t,\tau)$ is only a function t, and is given by

$$A(t) = F^*(t,t)F^{-1}(t,t) \tag{5.23}$$

It may be shown similarly that $C_1(t,\tau)$ is not a function of τ, nor is $B(t,\tau)$ a function of t. That is,

$$C_1(t) = F_1(t,t)F^{-1}(t,t) \tag{5.24}$$

$$B_1(\tau) = F^{-1}(\tau,\tau)F_2(\tau,\tau) . \tag{5.25}$$

Following directly from (5.23), as in the single-input-single-output case, we obtain the separated form of $F(t, \tau)$,

$$F(t, \tau) = F(t, \lambda) F^{-1}(\tau, \lambda) F(\tau, \tau). \tag{5.26}$$

Thus,

$$H(t, \tau) = C_1(t) F(t, \lambda) F^{-1}(\tau, \lambda) F(\tau, \tau) B_1(\tau) , \tag{5.27}$$

and is realizable by the n^{th} order system (A, B, C), where A(t) is given by (5.23) and

$$B(t) = F_2(t, t) \tag{5.28}$$

$$C(t) = F_1(t, t) F^{-1}(t, t). \tag{5.29}$$

Since $\ell \leq n$ and $k \leq n$, $\Gamma_{nn}(t, \tau)$ must have rank n for all $t \geq \tau$. Therefore, $\Gamma_{nn}(t, t)$ has rank n for all t which in turn implies $Q_0(t)$ and $Q_c(t)$ must have rank n for all t. The system (A, B, C) is thus an n^{th} order uniformly observable and controllable realization of $H(t, \tau)$. Furthermore, as in the scalar case, the matrices A, B and C will be constant if $H(t, \tau) = H(t - \tau)$.

An important feature of this realization procedure is that the systems obtained will always be <u>minimal</u>. A system is said to be

a minimal realization of an impulse response matrix $H(t,\tau)$ if there is no system of lower order which also realizes $H(t,\tau)$. It is clear from our earlier discussion of reducibility that a minimal realization is completely controllable and observable (globally reduced [5]). The converse is not generally true, however, as shown by Desoer [6] . For uniformly (and totally) controllable and observable systems the converse does hold, as will now be demonstrated.

Theorem 5. 3: A totally controllable and totally observable system is a minimal realization of its impulse response matrix.

Let (A, B, C) be a totally controllable and totally observable system of order n, and suppose that a system $(\bar{A}, \bar{B}, \bar{C})$ of order $n-1$ has the same impulse response matrix as (A, B, C). The impulse response matrix can then be written in two forms:

$$(1) \qquad H(t,\tau) = \Psi(t)\Theta(\tau) \quad , \qquad t \geq \tau$$

where $\Psi(t)$ and $\Theta(\tau)$ have n columns and n rows respectively.

$$(2) \qquad H(t,\tau) = \bar{\Psi}(t)\bar{\Theta}(\tau) \quad , \qquad t \geq \tau$$

where $\bar{\Psi}(t)$ and $\bar{\Theta}(\tau)$ have $n-1$ columns and $n-1$ rows respectively.

If $\Gamma(t,\tau)$ is calculated from (2), it clearly has rank $< n$ for all t and τ. From (1), however,

$$\Gamma(t,\tau) = W_0(t)\, W_c(\tau) \qquad t \geq \tau$$

where both $W_0(t)$ and $W_c(t)$ do not have rank $<n$ on any interval. Consequently, there exist values of τ and t with $t \geq \tau$ for which $\Gamma(t,\tau)$ has rank n. This contradiction establishes the theorem.

Since the system obtained by the above realization procedure is uniformly controllable and observable, Theorem 5.3 implies it must be a minimal realization of $H(t,\tau)$. Furthermore, it is unique within an equivalence transformation as shown by:

Theorem 5.4:* If $(A, B, C) \in U_n$ is a realization of $H(t,\tau)$, then any other minimal realization is equivalent to (A, B, C).

Let $(\bar{A}, \bar{B}, \bar{C})$ be any n^{th} order realization of $H(t,\tau)$. If Q_0^+, Q_c^+ and $\bar{Q}_0^+$, $\bar{Q}_c^+$ are the identification matrices of (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$, respectively, then clearly

$$(Q_0^+(t))' Q_c^+(t) = \Gamma_{n+1,\, n+1}(t, t) = (\bar{Q}_0^+(t))' \bar{Q}_c^+(t).$$

Thus, it follows from Theorem 4.2 that $(\bar{A}, \bar{B}, \bar{C})$ is equivalent to (A, B, C). The transformation relating the two systems is as given in Theorem 4.2 .

---

*Theorem 5.4 also follows from Youla ([5], Corollary 1).

5.4 Synthesis of Fixed Systems

The response of a fixed system may be specified by its impulse response matrix $H(t - \tau)$ or its transfer function matrix $G(s)$ (the Laplace transform of $H(t)$). Realizability of $H(t)$ or $G(s)$ is easily determined by examining the form of each element of the matrices [2]. For example, $G(s)$ is realizable if and only if all of its elements are the ratios of polynomials with the dominator polynomial of higher degree than the numerator polynomial (i.e., a proper rational matrix). Finding minimal realizations of a given $H(t)$ or $G(s)$ is far from a trivial problem, however, and has been the subject of considerable research in recent years [1, 2, 3, 5, 6]. Several procedures have been offered for constructing minimal realizations and for determining their order, but they are in general quite complex when $G(s)$ has repeated poles.

We will present below, a new approach to the realization problem for fixed systems by specializing the results of the previous section. This method has the advantage of yielding simultaneously, an explicit measure of the degree of the minimum order realizations of $H(t)$ and a straightforward synthesis procedure which in no way depends on the type of poles possessed by $G(s)$.

Let $H(t)$ be an $m \times r$ matrix function of t. Certainly, if $H(t)$ is to be the impulse response matrix of a fixed system it must be continuously differentiable an arbitrary number of times. Thus

we may define for all i , j

$$\Gamma_{ij}(t) = \begin{bmatrix} H(t) & H^{(1)}(t) & \cdots\cdots & H^{(j-1)}(t) \\ H^{(1)}(t) & H^{(2)}(t) & \cdots\cdots & H^{(j)}(t) \\ \vdots & \vdots & & \vdots \\ H^{(i-1)}(t) & H^{(i)}(t) & \cdots\cdots & H^{(i+j-2)}(t) \end{bmatrix}$$

where

$$H^{(i)}(t) = \frac{d^i}{dt^i} H(t) \quad .$$

Save for the sign of alternate columns, $\Gamma_{ij}(t)$ is equivalent to $\Gamma_{ij}(t, 0)$ as defined in the previous section. As the sign changes do not effect the rank of any of the matrices, it is more convenient to use the present definition.

The realizability criteria for fixed systems may now be given. The proof closely follows that of Theorem 5. 2.

Theorem 5. 5: $H(t)$ is the impulse response matrix of a fixed completely controllable and completely observable system of order n if and only if there exist non-negative integers $\ell \leq n$ and $k \leq n$ such that the matrices $\Gamma_{\ell k}(t)$ and $\Gamma_{\ell+1, k+1}(t)$ have rank n for all $t \geq 0$ and an $n \times n$ submatrix of $\Gamma_{\ell k}(t)$ has rank n for all $t \geq 0$.

Let $F(t)$ be an $n \times n$ submatrix of $\Gamma_{\ell,k}(t)$ having rank n for all $t \geq 0$. With respect to this matrix, let the matrices $F^*(t)$, $F_1(t)$ and $F_2(t)$ be defined precisely as in the proof of Theorem 5.2. Then,

$$H(t) = C_1(t) F(t) B_1(t) \tag{5.30}$$

where

$$C_1(t) = F_1(t) F^{-1}(t) \tag{5.31}$$

and

$$B_1(t) = F^{-1}(t) F_2(t) . \tag{5.32}$$

Also,

$$F^*(t) = F^{(1)}(t) = A(t) F(t) , \tag{5.33}$$

since both $\Gamma_{\ell+1,k+1}(t)$ and $\Gamma_{\ell,k}(t)$ have rank n . Furthermore, $A(t)$ is not a function of t since

$$\frac{d}{dt} A(t) = F^{(2)}(t) F^{-1}(t) - F^{(1)}(t) F^{-1}(t) F^{(1)}(t) F^{-1}(t) ,$$

and

$$F^{(2)}(t) = A(t)F^{(1)}(t)$$

imply that

$$\frac{d}{dt}A(t) = 0 .$$

It may be shown similarly that $C_1(t)$ and $B_2(t)$ are constant matrices.

Thus,

$$\dot{F}(t) = AF(t) , \qquad (5.34)$$

and

$$H(t) = C_1 F(t) B_1 \qquad (5.35)$$

Since $F(t)$ has rank n for all t , (5.34) implies that

$$F(t) = e^{At}F(0)$$

and is a fundamental matrix for the system of n differential equations

$$\dot{x} = Ax .$$

From equations (5.31), (5.32) and (5.34)

$$A = \dot{F}(0)F^{-1}(0) \tag{5.36}$$

$$B_1 = F^{-1}(0)F_2(0) \tag{5.37}$$

$$C_1 = F_1(0)F^{-1}(0) . \tag{5.38}$$

Thus,

$$H(t) = F_1(0)F^{-1}(0)e^{At}F_2(0) \tag{5.39}$$

and is realized by the n^{th} order system (A, B, C) where

$$B = F_2(0) \tag{5.40}$$

and

$$C = F_1(0)F^{-1}(0) . \tag{5.41}$$

Furthermore, if

$$Q_c = [\, B \vdots AB \vdots \; . \; . \; . \vdots A^{n-1}B]$$

and

$$Q_0 = [\,C' \,\vdots\, A'C' \,\vdots\, \ldots \,\vdots\, (A^{n-1})'C'\,]$$

then

$$\Gamma_{nn}(0) = Q_0'Q_c \ .$$

Since $\Gamma_{nn}(0)$ has rank n, (A, B, C) is a completely controllable and observable system.

The converse of Theroem 5.5 is easily established. If (A, B, C) is an n^{th} order realization of H(t) then it may be verified that

$$\Gamma_{nn}(t) = Q_0'e^{At}Q_c \ .$$

If Q_0' and Q_c have rank n, then certainly $\Gamma_{nn}(t)$ has rank n for all t. Similarly,

$$\Gamma_{n+1,n+1}(t) = (Q_0^+)'e^{At}Q_c^+$$

has rank n for all t. Also, if Q_c and Q_0' have rank n they must contain $n \times n$ submatrices, F_c and F_0, respectively, having rank n.

Consequently,

$$F(t) = F_0 e^{At} F_c$$

is an n x n submatrix of $\Gamma_{nn}(t)$ having rank n for all t .

It follows immediately from Theorem 5. 3 that the system (A, B, C) derived above is a minimal realization of H(t) and from Theorem 5. 4 it is assured that (A, B, C) is unique within an equivalence transformation. Furthermore, Theorem 4.1 implies that all constant minimal realizations of H(t) are related to (A, B, C) by a constant transformation.

Example 5. 2: Consider the matric function

$$H(t) = \begin{bmatrix} e^{-t} & e^{-t}(8+3t) \\ 2e^{-t} & e^{-t}(4+6t) \end{bmatrix} .$$

The determinant of H(t) is $-12e^{-t}$, so that $\Gamma_{11}(t) = H(t)$ has rank 2 for all t . $\Gamma_{22}(t)$ is found to be

$$\Gamma_{22}(t) = \begin{bmatrix} e^{-t} & e^{-t}(8+3t) & -e^{-t} & -e^{-t}(5+3t) \\ 2e^{-t} & e^{-t}(4+6t) & -2e^{-t} & -e^{-t}(-2+6t) \\ -e^{-t} & -e^{-t}(5+3t) & e^{-t} & e^{-t}(2+3t) \\ -2e^{-t} & -e^{-t}(-2+6t) & 2e^{-t} & e^{-t}(-8+6t) \end{bmatrix}$$

Denote the columns of $\Gamma_{22}(t)$ by g_1, g_2, g_3 and g_4 , then it is clear

that $g_3 = -g_1$ and $g_4 = 3g_1 - g_2$, so that $\Gamma_{22}(t)$ also has rank 2 for all t. Let $F(t) = H(t)$; then

$$F_1(t) = F_2(t) = F(t)$$

and

$$\dot{F}(t) = \dot{H}(t) .$$

Therefore,

$$F(0) = \begin{bmatrix} 1 & 8 \\ 2 & 4 \end{bmatrix} = F_1(0) = F_2(0)$$

and

$$\dot{F}(0) = \begin{bmatrix} -1 & -5 \\ -2 & 2 \end{bmatrix} .$$

Thus $H(t)$ is realized by the second order completely controllable and observable system (A, B, C), where

$$A = -\frac{1}{4}\begin{bmatrix} 2 & 1 \\ -4 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 8 \\ 2 & 4 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Although the above procedure provides a self-contained theory for minimal realization of fixed impulse response and transfer function matrices, there are several alternate methods which might be better suited for synthesizing high order systems on a digital computer. Among these are Gilbert's method for transfer function matrices with simple poles [1], and Desoer's extention for multiple poles [6]. The advantage of these methods for fixed systems is that they do not require operations on time-variable matrices. We will describe another such computational procedure below which utilizes the explicit method of system reduction presented in Chapter 4.

As observed in the discussion prior to Theorem 5.1, one can always determine by inspection the order (say $\bar{n}$) of some realization of a fixed impulse response, or transfer function matrix. Kalman [2] uses this information in a synthesis procedure which entails first constructing a realization of order $\bar{n}$, and then reducing it to minimal order.

A modification of this approach, which will now be outlined, greatly reduces the computation required and results in an explicit form for a minimal realization of $G(s)$. This method is based on the procedure given in Chapter 4 for obtaining a least order representation of a system from its Υ matrix.

(1) Determine $\bar{n}$, an upper bound on the order of the minimal realizations, by inspection of $G(s)$ (one such method for finding $\bar{n}$ is given by Kalman [2].

(2) Form $\Gamma_{\bar{n}\bar{n}}(0)$. It is clear that the elements of $\Gamma_{\bar{n}\bar{n}}(0)$ can be computed directly from $G(s)$; that is,

$$H(0) = \lim_{s \to \infty} s\, G(s) \quad ,$$

$$H^{(1)}(0) = \lim_{s \to \infty} s[\, s\, G(s) - H(0)] \quad ,$$

etc.

(3) Compute the rank n of $\Gamma_{\bar{n}\bar{n}}(0)$. From Theorem 4. 1, n is the order of the minimal realizations of $G(s)$.

(4) Let Υ be the submatrix formed from the first $m(n+1)$ rows and the first $r(n+1)$ columns of $\Gamma_{\bar{n}\bar{n}}(0)$ (i. e., $\Upsilon = \Gamma_{n+1,\, n+1}(0)$). An explicit minimal realization of $G(s)$ is obtained from Υ as in Theorem 4. 3.

The method for realizing fixed transfer function matrices is quite similar to those recently presented by Ho and Kalman [11], and by Youla and Tissi [12] while the present work was in progress. In both [11] and [12], a sequence of matrices equivalent to $\Gamma_{kk}(0)$ is generated from G(s) but the methods of obtaining a minimal realization from these matrices differ in several aspects from that presented here.

References

[1] E. G. Gilbert, "Controllability and Observability in Multi-Variable Control Systems," J. SIAM Control, vol. 1, pp. 128-151, 1963.

[2] R. E. Kalman, "Mathematical Description of Linear Dynamical Systems," J. SIAM Control, vol. 1, pp. 152-192, 1963.

[3] R. E. Kalman, "Irreducible Realizations and the Degree of a Rational Matrix," J. SIAM, vol. 13, pp. 520-545, June, 1965.

[4] L. Weiss and R. E. Kalman, "Contributions to Linear System Theory," Technical Report 64-9, Research Institute for Advanced Study, Maryland; March, 1964.

[5] D. C. Youla, "The Synthesis of Linear Dynamical Systems from Prescribed Weighting Patterns," PIB Report MRI-1271-65, Department of Electrophysics Polytechnic Institute of Brooklyn, June 1965.

[6] C. A. Desoer, "The Minimal Realization of Impulse Response Matrices," ERL Tech. Memorandum M-128, University of California, Berkeley; September, 1965.

[7] S. Darlington, "An Introduction to Time-Variable Networks," Proc. Symp. on Circuit Analysis, University of Illinois, Urbana, 1955.

[8] S. Darlington, "Nonstationary Smoothing and Prediction Using Network Theory Concepts," IRE Trans. on Circuit Theory (special suppl.) vol. CT-6, pp. 1-11, May, 1959.

[9] A. M. Batkov, "The Problem of Synthesizing Linear Dynamical Systems with Variable Parameters," Automation and Remote Control, vol. 19, pp. 42-47; January, 1958.

[10] V. Borskii, "On the Properties of Impulsive Response Functions of Systems with Variable Parameters," Automation and Remote Control, vol. 20, pp. 822-830; July, 1959.

[11] B. L. Ho and R. E. Kalman, "Effective Construction of Linear State-Variable Models from Input/Output Data," Proc.

Allerton Conf. on Circuit and System Theory, University of Illinois, Urbana, pp. 449-459; October, 1965.

[12] D. C. Youla and Plinio Tissi, "n-Port Synthesis via Reactance Extraction-Part V," IEEE Conv. Record, vol. 14, part 7, pp. 183-208; 1966.

CHAPTER 6

CONTROLLABILITY AND TIME-VARIABLE UNILATERAL NETWORKS*

6.1 Introduction

It has been seen in the previous chapters that the concepts of controllability and observability are valuable characterizing properties in the general theory of linear systems. Although this viewpoint has not received much explicit attention by circuit theorists, it appears that significant problems in network theory might be formulated clearly and solved effectively through arguments utilizing controllability [1, 2, 3, 4].

The field of time-variable network theory could especially benefit from the introduction of new techniques since relatively little progress has been made in this field by comparison with the accomplishments of fixed network theory. These considerations motivate a closer examination of controllability criteria as applied to variable-parameter systems and an attempt to exploit these criteria in a specific application to time-variable network theory.

To demonstrate the applicability of the controllability concept to network theory, it is used to develop a class of networks, composed of only linear two-terminal elements, which exhibit unilateral transmission between ports. The fact that such networks exist does not appear to be well known. Although it is felt that workers in the field

---

*Parts of this chapter have appeared in a paper by the author and H. E. Meadows [8].

of sampled-data systems may be aware of the possibility, to the author's knowledge only one example of this type of network has been presented in the literature. R. J. Mohr [5] recently described a circuit containing delay lines and periodic switches which behaves essentially as a three-port circulator. Such unilateral networks have not been studied in any generality, however.

In this paper the normal form of the network equations is used to show that unilateral behavior and controllability are related and that the conditions for controllability developed for general time-variable systems can be used to characterize classes of unilateral networks. These networks must contain time-variable components. An example of a particular class of unilateral networks is presented in the subsequent discussion.

6. 2 Output Controllability

It will be shown subsequently that unilateral behavior in a network is intimately related to the concept of output controllability introduced by Kreindler and Sarachik [6]. Several types of output controllability may be defined (as was done for state controllability in Chapter 2), but the following will suffice for our purpose here.

Definition 6. 1: System (2. 1) is completely output controllable at t_0, if for any initial state x_0 at t_0, and any desired final output y_d, there exists a finite time $t_1 > t_0$ and an input $u(t)$ defined on the interval $[t_0, t_1]$ such that $y(t_1) = y_d$.

The necessary and sufficient condition for this type of controllability established by Kreindler and Sarachik [6] is stated below.

Theorem 6.1: System (2.1) is completely output controllable at t_0 if there exists a finite time $t_1 > t_0$ such that the rows of $H(t_1, \tau)$ are linearly independent functions of τ on $[t_0, t_1]$.

In order that the concept of output controllability be applicable to the study of time-variable networks, it is necessary to obtain a criterion that does not depend on the impulse response matrix. This is accomplished below with the aid of the techniques employed in Chapter 3 for state controllability.

Theorem 6.2: System (2.1) is completely output controllable at t_0 if for some $t > t_0$ the matrix $C(t)Q_c(t)$ has rank m.

To prove this theorem, let $W(t, \tau)$ be the Wronskian matrix of the rows of $H(t, \tau)$, that is

$$W(t,\tau) = [\Lambda_0(t,\tau) \vdots \Lambda_1(t,\tau) \vdots \ldots \vdots \Lambda_{n-1}(t,\tau)] ,$$

where

$$\Lambda_k(t,\tau) = \frac{\partial^k}{\partial\tau^k} H(t,\tau); \quad \Lambda_0(t,\tau) = H(t,\tau) .$$

From (3.8) it is clear that

$$W(t,\tau) = C(t)\Phi(t,\tau)Q_c(\tau) .$$

Suppose that the system is not completely output controllable. Then, Theorem 6.1 implies that for all $t > t_0$, the rows of $H(t,\tau)$ are linearly dependent functions of τ on $[t_0, t]$. It follows then from Theorem 3.1 that $W(t,\tau)$ has rank $<m$ for all t, and for all $\tau \epsilon [t_0, t]$. In particular,

$$W(t,t) = C(t)Q_c(t)$$

has rank $<m$ for all t, which establishes the theorem.

6.3 Unilateral Time-Variable Networks

In the preceding discussion of controllability, the particular type of physical system represented by the system of differential equations was not considered. The present section is concerned with the system representation of linear electrical networks; in particular, the network property of unilateral transmission and the system property of controllability are related. The sufficient conditions for controllability are used to demonstrate the existence of a class of unilateral time-variable networks composed of lumped, linear, two-terminal elements. The controllability argument ends insight and some generality to this demonstration while also obviating the integration of time-variable differential equations.

The very well-known concept of unilateral transmission is a simple one which may be stated in terms of the impulse response

associated with a pair of ports. Consider a network N containing only linear two-terminal elements, and let i and j be any two ports of N. Denote by $h_{ij}(t, \tau)$ the voltage [current], response at port i and time t to a unit impulse of current [voltage] at port j and time τ. Then the network is a reciprocal network if $h_{ji}(t, \tau) = h_{ij}(t, \tau)$ for all t, τ and all i, j. It is well known that unless all of its elements are fixed, N generally may be nonreciprocal. The existence of nonreciprocal effects suggests the possibility of completely unilateral behavior in time-variable networks. If $h_{ij}(t, \tau) = 0$ for all t and τ, and if for some t and τ, $h_{ji}(t, \tau) \neq 0$, N is unilateral from port i to port j. Networks of linear two-terminal elements exhibiting unilateral transmission between ports appear not to have been examined in any generality.

The main problem in studying this type of unilateral behavior is that the networks exhibiting it must contain time-variable elements. Since the exact behavior of a time-variable network is difficult to determine, it seems useful to develop techniques which do not require knowledge of network solutions. If the normal form (A-matrix description) is used to write the network equations, the unilateral property may be phrased in terms of controllability of a linear system and the test given in the preceding section may then be employed to bypass the system solution.

The equilibrium equations of a linear lumped RLC network may be written in the normal form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{6.1}$$

as in (2.1). For convenience, and with some loss of generality, it will be assumed that the input and output ports are chosen so that the n vector $x(t)$ represents both the state variables (e.g., capacitor voltages for an RC network) and the port outputs, and the n vector $u(t)$ represents the inputs to the same ports (e.g., current sources in shunt with ports of an RC network). The work of Bryant [7] may be used to determine the matrices $A(t)$ and $B(t)$ explicitly.

To deal with transmission between pairs of ports of the network, it is convenient to recast (6.1) in a slightly modified form. Let e_i represent the unit column vector with a one in the i^{th} position and zeros elsewhere, and e_i' the corresponding unit row vector; then in order to consider inputs applied only at port i and outputs only at port j, (6.1) may be replaced by the system F_{ji} described by

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)e_i u(t) \\ y(t) &= e_j' \, x(t) \end{aligned} \tag{6.2}$$

<u>Theorem 6.3:</u> The network with normal form equations (6.2) is unilateral from i to j if and only if system F_{ij} is nowhere controllable, and system F_{ji} is completely output controllable at some time t_0.

By "nowhere controllable" we mean that the system is not completely output controllable at any time. The theorem is clear since $h_{ji}(t,\tau)$ is simply the impulse response of F_{ji} ; that is,

$$h_{ji}(t,\tau) = e_j' \Phi(t,\tau)B(\tau)e_i .$$

To establish that networks satisfying the criteria of Theorem 6.3 do exist, a class of RC unilateral networks will be derived. Provided that $C^{-1}(t)$ and $\dot{C}(t)$ exist for all t , the equilibrium equations for an RC network can be written as

$$\dot{V}(t) + C^{-1}(t)[\dot{C}(t) + G(t)]V(t) = C^{-1}(t)J(t) , \qquad (6.3)$$

where $V(t)$ represents a set of voltage variables pertinent to a tree in a capacitive subgraph of the network, and $C(t)$ and $G(t)$ are the n-port capacitance and conductance matrices corresponding to the voltage variables. The current vector $J(t)$ represents a set of current sources in parallel with the ports determined by the voltage variables. Thus, the order of the system (6.3), is the same as the order of complexity of the network, and the elements of $V(t)$ and $J(t)$ represent output voltage and input currents, respectively. It will now be shown that a network with system equation (6.3) may be unilateral. Define $S(t) = C^{-1}(t)$; then system F_{ij} corresponding to (6.3) is

$$\dot{V}(t) = -S(t)[\dot{C}(t) + G(t)]\, V(t) + S(t)e_j j_j(t) \tag{6.4}$$

$$v_i(t) = e_i' V(t) \; .$$

A necessary condition for F_{ij} to be nowhere controllable is that the ij element of $S(t)$ must vanish identically. If the contrary were true then $h_{ij}(t,\tau) = 0$ would imply that, for all t and for all τ such that $s_{ij}(\tau) \neq 0$,

$$\varphi_{ii}(t,\tau) = \frac{1}{s_{ij}(\tau)} \sum_{\substack{k=1 \\ k\neq j}}^{n} \varphi_{ik}(t,\tau) s_{kj}(\tau) \; . \tag{6.5}$$

But,

$$\varphi_{ik}(\tau,\tau) = \begin{cases} 0 \, , & k \neq i \\ 1 \, , & k = i \, ; \end{cases}$$

therefore, (6.5) is contradictory at $t = \tau$ and $s_{ij}(\tau)$ must be zero.

Consider now the case $n = 3$, and let $i = 3$ and $j = 1$. Since the port matrices $C(t)$ and $G(t)$ are symmetric,

$$s_{13}(t) = s_{31}(t) = 0 \; .$$

With this constraint, the condition for F_{13} to be output controllable at some t_0 (Theorem 6.2) is that the matrix

$$e_1'[\,0 \,\vdots\, -A(t)S(t)e_3 \,\vdots\, \{A^2(t)S(t) - 2A(t)\dot{S}(t) - \dot{A}(t)S(t)\}e_3] \qquad (6.6)$$

have rank 1 for some $t_1 > t_0$. Moreover, F_{31} is nowhere controllability if and only if

$$h_{31}(t,\tau) = \varphi_{31}(t,\tau)s_{11}(\tau) + \varphi_{32}(t,\tau)s_{12}(\tau) = 0 \qquad (6.7)$$

To simplify the presentation without great loss of generality, let

$$s_{12}(t) = s_{21}(t) = 0\ ;$$

then (6.7) becomes

$$\varphi_{31}(t,\tau)s_{11}(\tau) = 0\ . \qquad (6.8)$$

The $A(t)$ matrix for system (6.4) can now be written as

$$- \begin{bmatrix} s_{11}(\dot{c}_{11} + g_{11}) & s_{11}g_{12} & s_{11}g_{13} \\ s_{22}g_{12} + s_{23}g_{13} & s_{22}(\dot{c}_{22} + g_{22}) + s_{23}(\dot{c}_{23} + g_{23}) & s_{22}(\dot{c}_{23} + g_{23}) + s_{23}(\dot{c}_{33} + g_{23}) \\ s_{23}g_{12} + s_{33}g_{13} & s_{23}(\dot{c}_{22} + g_{22}) + s_{33}(\dot{c}_{23} + g_{23}) & s_{23}(\dot{c}_{23} + g_{23}) + s_{33}(\dot{c}_{33} + g_{33}) \end{bmatrix},$$

where the dependence of each coefficient on t is to be understood.

If $s_{11}(t) = 0$, it is clear from the A matrix that $\varphi_{12}(t,\tau)$ and $\varphi_{13}(t,\tau)$ are always zero. But this would imply that

$$h_{13}(t,\tau) = \varphi_{13}(t,\tau)s_{23}(\tau) + \varphi_{13}(t,\tau)s_{33}(\tau) = 0\,,$$

or F_{13} is nowhere controllable. Therefore, (6.8) implies that

$$\varphi_{31}(t,\tau) = 0\,. \tag{6.9}$$

But $\varphi_{31}(t,\tau) = 0$ if $a_{31} = 0$ and either $a_{21} = 0$ or $a_{32} = 0$. It will now be shown that the latter choice preserves the controllability of F_{13}. If $a_{31} = a_{32} = 0$, then (6.6) reduces to

$$[\,0 \,\vdots\, 0 \,\vdots\, s_{11}(g_{12}\dot{s}_{23} + g_{13}\dot{s}_{33})]\,,$$

so that F_{13} is output controllable if $g_{12}\dot{s}_{23} \neq -g_{13}\dot{s}_{33}$.

A set of constraints can now be summarized that will insure that a third-order RC network is unilateral:

$$c_{12} = c_{13} = 0\,, \tag{6.10a}$$

$$\dot{c}_{22} = -g_{22}, \qquad \dot{c}_{23} = -g_{23}\,, \tag{6.10b}$$

$$g_{13} = \frac{c_{23}}{c_{22}} g_{12} \neq 0 \, , \tag{6.10c}$$

$$c_{22}\dot{c}_{23} \neq \dot{c}_{22}c_{23} \, . \tag{6.10d}$$

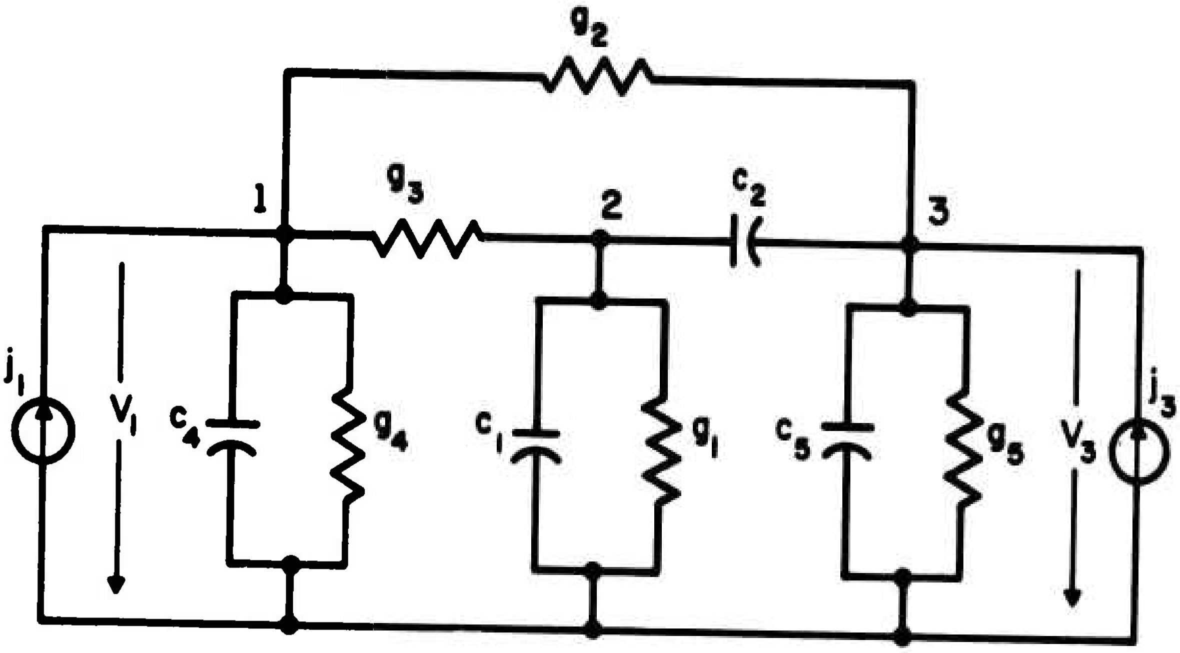
An illustrative network is provided by the following example.

Example 6.1: Consider the circuit of Fig. 6.1, whose port matrices are

$$G = \begin{bmatrix} 1 & 4 + \cos t & -\left(\frac{4 + \cos t}{5 + \cos t}\right) \\ 4 & \sin t & 0 \\ -\left(\frac{4 + \cos t}{5 + \cos t}\right) & 0 & 1 \end{bmatrix} ,$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 + \cos t & -1 \\ 0 & -1 & 2 \end{bmatrix} .$$

Examination of G and C shows that the constraints in (6.10) are satisfied, so that the network transmits from port 3 to port 1, but not in the opposite direction. It may be observed that the constraints in (6.10) are independent of c_{11}, g_{11}, c_{33}, and g_{33}. Therefore, ports 1 and 3 could be padded arbitrarily while preserving the unilateral trans-



$c_1 = 4 + \cos t$ $\qquad c_2 = c_4 = c_5$

$g_1 = 4 + \cos t$ $\qquad g_2 = \dfrac{4 + \cos t}{5 + \cos t}$

$g_3 = -(4 + \cos t)$ $\qquad g_4 = 1 + \dfrac{(4 + \cos t)^2}{5 + \cos t}$

$$g_5 = \frac{1}{5 + \cos t}$$

Fig. 6.1 A unilateral RC network

mission property, as would be expected intuitively. Several other examples of RC unilateral networks are given in [8].

The network of Example 6.1 contains one negative element, but as will be shown below, the circuit behaves in a stable manner under all excitations. This analysis will also contribute further physical insight into the unilateral nature of the network.

Consider the network of Example 6.1 with the elements in parallel with ports 1 and 3 removed as shown in Fig. 6.2. This simplified circuit may also be represented as in Fig. 6.3.

It may be readily verified that if the network is excited by j_1, then

$$\dot{v}_2 = -v_2 + b(t)j_1 \tag{6.11}$$

and

$$\dot{v}_4 = -v_4 - b(t)j_1 , \tag{6.12}$$

where

$$b(t) = \frac{1}{4 + \cos t} .$$

It is clear that for any input j_1, the output voltage at port 3 is

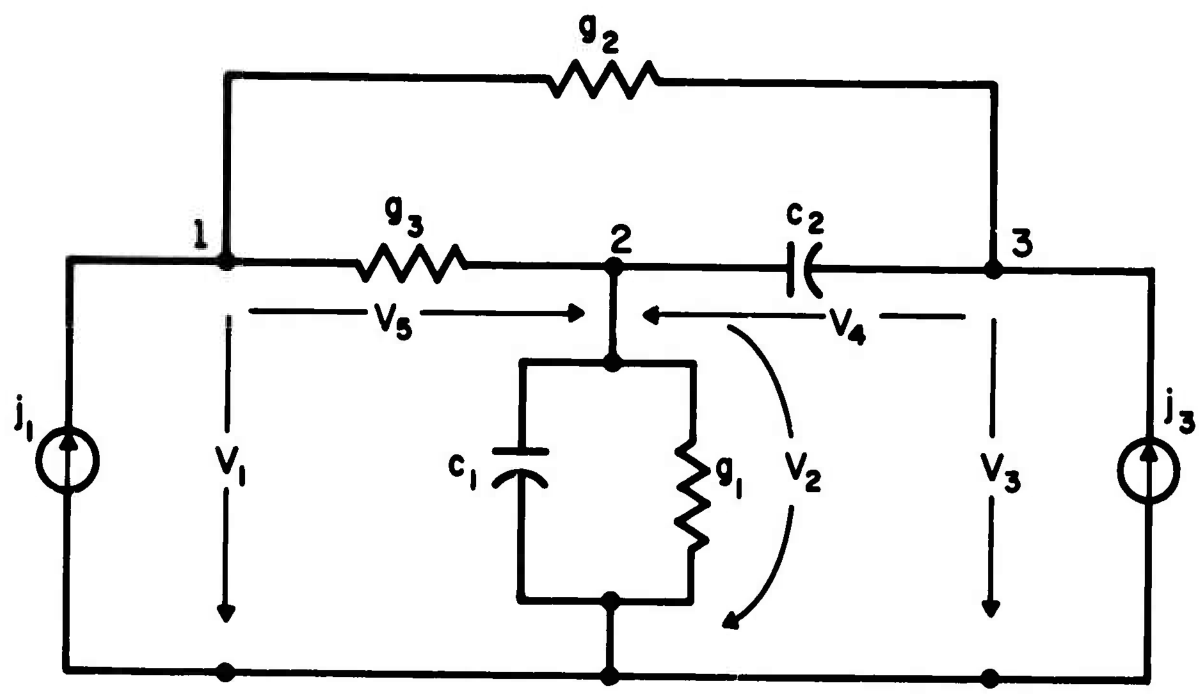


Fig. 6.2 A simplified unilateral network

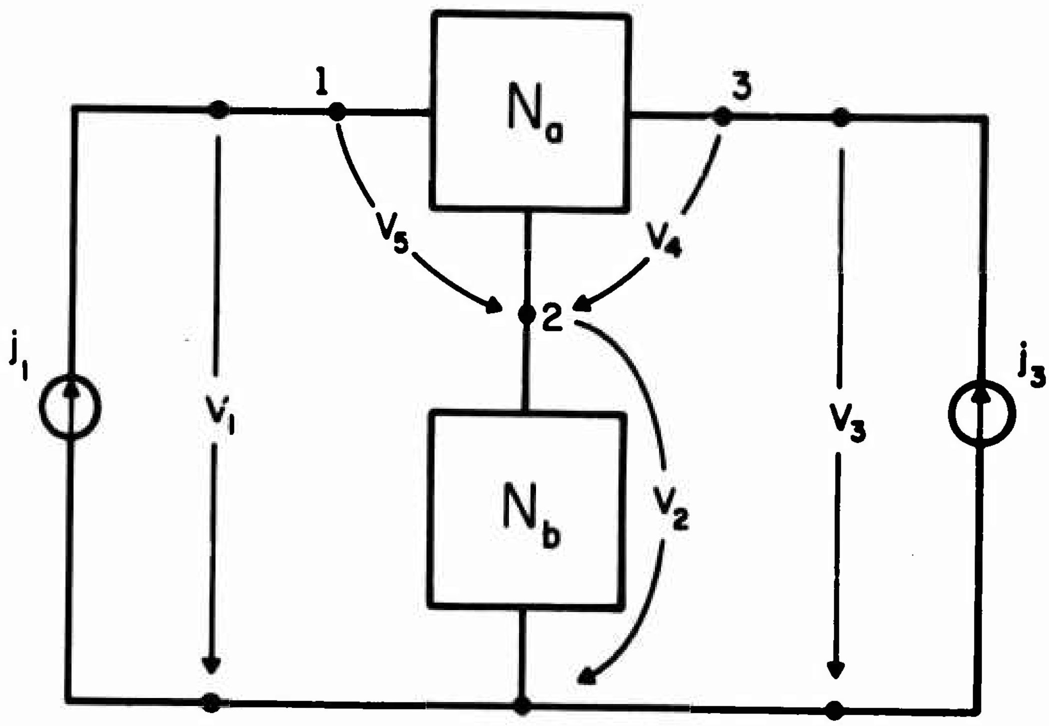


Fig. 6.3 Representation of a class of unilateral networks

identically zero for zero initial conditions, since

$$v_3 = v_2 + v_4 = 0 . \tag{6.13}$$

If the network is excited by j_3, however, this type of voltage cancellation does not take place at port 1 since N_a is a nonreciprocal network, as will now be shown. The response at port 1 of N_a is given by

$$\begin{aligned} \dot{v}_4 &= -v_4 + j_3 \\ v_5 &= -bv_4 \end{aligned} \tag{6.14}$$

or

$$\dot{v}_5 = -(1 - \frac{\dot{b}}{b})v_5 - bj_3 , \tag{6.15}$$

while that at port 2 of N_b is the same as (6.11) with j_1 replaced by j_3. The nonreciprocal nature of N_a is evidenced by the term $\dot{b}/b$. If b were constant, (6.15) would be identical with (6.12). Thus,

$$v_1 = v_4 + v_5 \neq 0 .$$

The impulse response in the direction of non-zero transmission is easily found, and is given by

$$h_{13}(t,\tau) = e^{-(t-\tau)}b(\tau) - b(t)e^{-(t-\tau)} . \qquad (6.16)$$

It is clear from (6.16) that the above network is stable in the sense that every bounded input $j_3(t)$ will produce a bounded output $v_1(t)$ [9], since for all t

$$\frac{1}{2} < b(t) < 1 .$$

Equations (6.11) and (6.12) are similarly stable.

The fact that unilateral transmission is possible in networks containing only time-variable RC elements indicates that many practical devices might be devised employing the nonreciprocal effects. Current investigations not yet completed have yielded configurations whose terminal behavior is equivalent to that of such basic circuits as gyrators, nullators, and norators [13]. Although these realizations can be shown to exist mathematically, many practical problems remain before they can be implemented physically. Chief among these are the feasibility of the required element variations, and the sensitivity of the unilateral transmission to perturbations in the elements.

Another possible area of application might be in the design of parametric amplifiers. In recent years several papers have dealt

with unilateral parametric amplifiers not requiring unilateral devices [10, 11, 12]. These amplifiers behave unilaterally only for fixed excitation frequency, however, and when viewed as linear circuits with their sources removed they do not have the unilateral property as defined here.

References

[1] R. E. Kalman, "On a New Characterization of Linear Passive Systems," Proc. 1st Allerton Conference on Circuit and System Theory, Univ. of Illinois; November, 1963.

[2] R. E. Kalman, "Irreducible Realizations and the Degree of of Rational Matrix," J. SIAM, vol 13, pp. 520-545.

[3] C. A. Desoer, "An Optimization Problem in Circuits," IEEE Trans. on Circuit Theory, vol. CT-12, pp. 28-31; March, 1965.

[4] D. C. Youla and P. Tissi, "n-Port Synthesis via Reactance Extraction-Part 1," IEEE Conv. Record, vol. 14, part 7, pp. 183-205, 1966.

[5] R. T. Mohr, "A New Nonreciprocal Transmission Line Device," Proc. IEEE (Correspondence), vol. 52, p. 612; May, 1964.

[6] E. Kreindler and P. E. Sarachik, "On the Concepts of Controllability and Observability of Linear Systems," IEEE Trans. on Automatic Control, vol. AC-9, pp. 129-136; April, 1964.

[7] P. R. Bryant, "The Explicit Form of Bashkow's A-Matrix," IRE Trans. on Circuit Theory, vol. CT-9, pp. 303-306; September, 1962.

[8] L. M. Silverman and H. E. Meadows, "Controllability and Time-Variable Unilateral Networks, "IEEE Trans. on Circuit Theory, vol CT-12, pp. 308-313; September, 1965.

[9] L. A. Zadeh and C. A. Desoer, Linear System Theory, McGraw-Hill, New York; 1963.

[10] H. B. Henning, "Non-Reciprocal Parametric Amplifiers: Theory and Instrumentation," IEEE Conv. Record, vol. 12, part 2, pp. 83-92; 1964.

[11] J. Klapper and M. Ghausi, "On the Properties of Combinations of Parametric Elements," IEEE Conv. Record, vol. 12, part 1, pp. 357-363; March, 1964.

[12] D. P. Howson, "A Parametric Converter Using Non- linear Resistence and Reactance," Proc. IEEE (Correspondence), vol. 53, pp. 1228-1229; September, 1965.

[13] H. J. Carlin and D. C. Youla, "Network Synthesis with Negative Resistors," Proc. IRE, vol. 49, pp. 907-920; May, 1961.

Chapter 7

CONCLUSION

In this thesis, a new characterization of several fundamental linear system concepts has been presented which employs only explicitly known system parameters and does not require the solution of time-variable differential equations. With this characterization, it was possible to provide a unified approach to the representation and synthesis of a broad class of variable parameter systems, and to derive an interesting class of time-variable networks. A detailed summary of the major results obtained is given below.

7.1 Summary of Results

It has been shown that the problem of determining the degree of controllability or observability of a time-variable linear system in terms of its coefficient matrix description is essentially equivalent to that of determining the degree of independence of a set of functions by a Wronskian type test. By developing such a test for vector functions, precise criteria for controllability and observability have been obtained which do not require the solution of time-variable differential equations. These criteria are based on the rank of the controllability and observability matrices, newly

defined quantities formed from the system coefficient matrices and a finite number of their derivatives. The controllability and observability matrices were also shown to be useful in a variety of other system analysis problems. In particular, a transformational property of these matrices was employed which enabled the formulation of general criteria for system reducibility and led to powerful methods for reducing noncontrollable or nonobservable systems to lower order. Also obtained were new methods for characterizing and generating equivalent system representations. Criteria for equivalence, zero-state equivalence, and zero-state time-invariance of time-variable linear systems were among the results of this development. Several useful canonical forms for time-variable linear systems were also investigated in this context and criteria for their existence and method for their construction were given.

Based on the theory of equivalent systems and the criteria for controllability and observability, a new approach to the synthesis of nonstationary impulse response matrices was developed. This method of synthesis, which does not require an a priori assumption of separability, provides a systematic procedure for realizing a wide class of system impulse responses. The specialization of this procedure to fixed systems is of independent interest, since it has several advantages in comparison with existing techniques.

By relating the concepts of controllability and unilateral transmission it was possible to use the controllability criteria to demonstrate the existence of a class of unilateral networks composed solely of two-terminal RC (time-variable) components. A stable example of such a network was presented and possible applications discussed.

7.2 Problems for Future Research

(1) It is well known [1, 2] that the optimum filter for determining the state variables of a system from observations of its output requires differentiators when some or all of the output measurements are assumed to be noise free. The precise conditions for existence of such a filter have not been given, however, nor the details of its construction. In the total absence of noise, equation (4.8) essentially provides the form of the filter, and it is clear that uniform observability is the required condition for existence. A detailed examination of the observability matrix should provide a method for constructing such filters and determining whether they exist under the more general noise constraints considered by Bryson and Johansen [2].

(2) The solution of many fixed control system problems is greatly simplified by first transforming a given system to an appropriate canonical form [3-6]. In fact, the conditions for existence of solutions to most of these problems coincide with the

conditions for existence of the canonical equivalents. It is expected, therefore, that the theory and methods of constructing canonical forms developed in Chapter 4 will prove to be valuable in extending previously restricted results to time-variable systems. For example, it can be shown with the aid of form (4. 33) that if a fixed single-input system is completely controllable, any desired pole pattern may be achieved by feeding back a linear combination of the state variables of the system. While the concept of a "pole" is somewhat nebulus for time-variable systems, it is clear that if all the coefficients $\bar{a}_i$ in (4. 33) can be adjusted independently by feedback, the system behavior can be modified as freely as in the fixed case. If a system is uniformly controllable, transformation to form (4. 33) is possible and the coefficients of the required feedback vector can be found precisely as in [3-4].

(3) It would be desirable to extend the synthesis procedure of Chapter 5 to more general impulse response matrices. While it seems unlikely that all linear systems can be realized by this method it should be possible to include realization of impulse response matrices of total controllable and observable systems. In addition, the use of the present method as part of an approximation procedure for realizing numerically tabulated response data (as would usually be specified in practice) should be investigated. A detailed evaluation of the various method for synthesizing fixed responses should also be

undertaken to determine the one most suitable for machine computation.

(4) As indicated in Chapter 6, there are many problems remaining before practical unilateral, and other nonreciprocal networks can be realized with practical time-variable components. If these problems can be overcome, however, a new method for building the non reciprocal elements widely used in network theory would be available.

References

[1] H. Cox, "Estimation of State Variables Via Dynamic Programming," Preprints Joint Automatic Control Conference, Stanford, Calif.; June, 1964.

[2] A. E. Bryson and D. E. Johansen, "Linear Filtering for Time-Varying Systems Using Measurements Containing Colored Noise," IEEE Trans. on Automatic Control, vol. AC-10, pp. 4-10; Jan., 1965.

[3] B. S. Morgan, "The Synthesis of Single Variable Systems by State Variable Feedback," Proc. Allerton Conf. on Circuit and System Theory, University of Illinois, Urbana; Nov., 1963.

[4] R. W. Brockett, "Poles, Zeros, and Feedback: State Space Interpretation," IEEE Trans. on Automatic Control, vol. AC-10, pp. 129-134; April, 1965.

[5] R. Sivan, "On Zeroing the Output and Maintaining It Zero," IEEE Trans. on Automatic Control, vol. AC-10, pp. 193-194; April, 1965.

[6] R. W. Bass and I. Gura, "High Order System Design Via State-Space Considerations," Preprints JACC, Troy, N. Y.; June, 1965.

APPENDIX

PROPERTIES OF ANALYTIC FUNCTIONS

The following well known properties of analytic functions are of importance here.

Property A. 1: If two analytic functions are equal at more than a finite number of points on a finite interval, they are equal everywhere on the interval. In particular, an analytic function that is zero at more than a finite number of points on a finite interval is identically zero on the interval.

Property A. 2: The sum and product of two analytic functions is an analytic function.

Property A. 3: If an analytic function is non zero on an interval, its inverse is analytic on the interval.

Property A. 4: The derivative of an analytic function is an analytic function.

The above properties of scalar analytic functions will now be utilized to derive several properties of matrices of analytic functions (analytic matrices).

Property A. 5: The determinant of any square submatrix of an analytic matrix is an analytic function.

Proof: Follows directly from Property A. 2.

Property A. 6: The rank of an analytic matrix is constant save possibly at a finite number of points on any finite interval.

Proof: Let $f(t)$ be the determinant of the largest square submatrix nonsingular for some point on a given finite interval. It follows from Properties A. 1 and A. 5 that $f(t)$ must be nonzero except possibly at a finite number of points on the interval.

Property A. 7: If an analytic matrix has rank q save possibly at a finite number of points on a finite interval then a square submatrix of order q also has this property.

Proof: Follows directly from Property A. 6.

It is now possible to prove Corollary 3. 2 of Chapter 3. First note that Property A. 4 implies that $W(t)$ is an analytic matrix if the row vector functions $\theta_1, \theta_2, \ldots, \theta_n$ are analytic. If $W(t)$ has rank n for some $t \in [t_0, t_1]$ then Theorem 3. 1 implies that the functions are independent on the interval. Suppose then that $W(t)$ has rank less than n for all $t \in [t_0, t_1]$, in which case it must have an essentially constant rank $q < n$ on the interval (Property A. 6). Let U be a subinterval of $[t_0, t_1]$ over which q rows of $W(t)$ have rank q everywhere (that such a subinterval exists, follows from Property A. 7). From Theorem 3. 2, there exists a constant nonsingular matrix T such that on U

$$TW(t) = \begin{bmatrix} \overline{W}(t) \\ 0 \end{bmatrix}, \qquad (A.1)$$

where $\bar{W}(t)$ has q rows and rank q. Since $TW(t)$ is analytic on the entire interval $[t_0, t_1]$ the relationship (A.1) must also hold on $[t_0, t_1]$ (Property A.1). Thus, as in Theorem 3.2 the θ_i are dependent over $[t_0, t_1]$ while exactly q of the θ_i are linearly independent on the interval (i.e., those corresponding to the rows of $W(t)$ having rank q on U).

ACKNOWLEDGEMENT

The author is deeply indebted to Professor Henry E. Meadows for the constant encouragement and advice he offered throughout the period of research leading to this thesis. Gratitude is also expressed to Professor Philip E. Sarachik of New York University and Dr. Bernard Friedland of the General Precision Company for many valuable discussions, to Professors Ralph T. Schwarz, Wan H. Kim and Robert E. King for their careful reading of the manuscript and helpful suggestions, and to colleagues Lane Brandenburg and Se Jeung Oh who listened patiently to early versions of the work and criticized freely.

This research was supported by the Office of Naval Research under Contract Number NON 4259(04) and by the National Science Foundation under Grant Number NSF GP-2789.

BIBLIOGRAPHY

[1] A. C. Aitken, Determinants and Matrices, Oliver and Boyd Ltd., London; 1939.

[2] M. Athans, "The Status of Optimal Control Theory and Applications for Deterministic Systems," IEEE Conv. Record, vol. 14, part 6, pp. 100-124; 1966.

[3] T. R. Bashkow, "The A Matrix, a New Network Description," IRE Trans. on Circuit Theory, vol. CT-4, pp. 117-119; Sept., 1957.

[4] R. W. Bass and I. Gura, " High Order System Design Via State-Space Considerations," Preprints JACC, Troy, N. Y.; June, 1965.

[5] A. M. Batkov, "The Problem of Synthesizing Linear Dynamical Systems with Variable Parameters," Automation and Remote Control vol. 19, pp. 42-47; January, 1958.

[6] W. R. Bennett, "A General Review of Linear Varying Parameter and Non-linear Circuit Analysis," Proc. IRE, vol. 38, pp. 259-263; March 1950.

[7] J. E. Bertram and P. E. Sarachik, "On Optimal Computer Control," Proc. 1st International Congress on Automatic Control, Buttersworths Scientific Publications, vol. 1, pp. 979-982, London; 1961.

[8] V. Borskii, "On the Properties of Impulsive Response Functions of Systems with Variable Parameters," Automation and Remote Control, vol. 20, pp. 822-830; July, 1959.

[9] R. W. Brockett, "Poles, Zeros, and Feedback: State Space Interpretation," IEEE Trans. on Automatic Control, vol. AC-10, pp. 129-134; April, 1965.

[10] P. R. Bryant, "The Explicit Form of Bashkow's A-Matrix, IRE Trans. on Circuit Theory, vol. CT-9, pp. 303-306; Sept., 1962.

[11] A. E. Bryson and D. E. Johansen, "Linear Filtering for Time-Varying Systems Using Measurements Containing Colored Noise," IEEE Trans. on Automatic Control, vol. AC-10, pp. 4-10; Jan., 1965.

[12] H. J. Carlin and D. C. Youla, "Network Synthesis with Negative Resistors," Proc. IRE, vol. 49, pp. 907-920; May, 1961.

[13] A. Chang, "An Algebraic Characterization of Controllability," IEEE Trans. on Automatic Control, vol. AC-10, pp. 112-113; January, 1965.

[14] M. R. Chidambara, "The Transformation to (Phase-Variable) Canonical Form," IEEE Trans. on Automatic Control, vol. AC-10, pp. 492-494; October, 1965.

[15] E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill Book Co., Inc., New York, N. Y.; 1955.

[16] H. Cox, "Estimation of State Variables Via Dynamic Programming," Preprints Joint Automatic Control Conference, Stanford, California; June, 1964.

[17] S. Darlington, "An Introduction to Time-Variable Networks," Proc. Symp. on Circuit Analysis, University of Illinois, Urbana; 1955.

[18] S. Darlington, "Nonstationary Smoothing and Prediction Using Network Theory Concepts," IRE Trans. on Circuit Theory (special suppl.), vol. CT-6, pp. 1-11; May, 1959.

[19] C. A. Desoer, "An Optimization Problem in Circuits," IEEE Trans. on Circuit Theory, vol. CT-12, pp. 28-31; March, 1965.

[20] C. A. Desoer, "The Minimal Realization of Impulse Response Matrices," ERL Tech Memorandum M-128, University of California, Berkeley; September, 1965.

[21] C. A. Desoer and A. Paige, "Linear Time Varying G-C Networks: Stable and Unstable," IEEE Trans. on Circuit Theory, vol. CT-10, pp. 180-190; June, 1963.

[22] R. V. Gamkrelidze, "Theory of Time-Optimal Processes for Linear System," Isv. Akad. Nauk., USSR, Ser. Mat., vol. 2, pp. 449-474

[23] E. G. Gilbert, "Controllability and Observability in Multi-Variable Control Systems," J. SIAM Control, vol. 1, pp. 128-151; 1963.

[24] H. B. Henning, "Non-Reciprocal Parametric Amplifiers: Theory and Instrumentation," IEEE Conv. Record, vol. 12, part 2, pp. 83-92; 1964.

[25] B. L. Ho and R. E. Kalman, "Effective Construction of Linear State-Variable Models from Input/Output Data," Proc. Allerton Conf. on Circuit and System Theory, University of Illinois, Urbana, pp. 449-459; October, 1965.

[26] D. P. Howson, "A Parametric Converter Using Non-linear Resistance and Reactance," Proc. IEEE (Correspondence), vol. 53, pp. 1228-1229; September, 1965.

[27] W. Hurewicz, Lectures on Ordinary Differential Equations, M.I.T. Press, Cambridge, Mass.; 1958.

[28] C. D. Johnson and W. M. Wonham, "A Note on the Transformation to Canonical (Phase-Variable) Form," IEEE Trans. on Automatic Control, vol. AC-9, pp. 312-313; July, 1964.

[29] C. D. Johnson and J. E. Gibson, "Optimal Control with Quadratic Performance Index and Fixed Terminal Time," IEEE Trans. on Automatic Control, vol. AC-9, pp. 355-360; October, 1964.

[30] R. E. Kalman, "On the General Theory of Control Systems," Proc. 1st Internat'l. Congress on Automatic Control, Buttersworths Scientific Publications, vol. 1, pp. 481-492, London, England; 1961.

[31] R. E. Kalman, "A New Approach to Linear Filtering and Prediction Problems," ASME Trans. (J. of Basic Eng.), vol. 82D, pp. 35-45; March, 1960.

[32] R. E. Kalman, "Canonical Structure of Linear Dynamical Systems," Proc. Nat'l. Acad. Sci., U.S., vol. 48, pp. 596-600; April, 1962.

[33] R. E. Kalman, "Mathematical Description of Linear Dynamical Systems," J. SIAM Control, vol. 1, pp. 152-192; 1963.

[34] R. E. Kalman, "On a New Characterization of Linear Passive Systems," Proc. 1st Allerton Converence on Circuit and System Theory, Univ. of Illinois; November, 1963.

[35] R. E. Kalman, "Lyapunov Functions for the Problem of Lurie in Automatic Controls," Proc. Nat. Acad. Sci., USA, vol. 49, pp. 201-205; 1963.

[36] R. E. Kalman, 'Irreducible Realizations and the Degree of a Rational Matrix," J. SIAM, vol. 13, pp. 520-545; June, 1965.

[37] R. E. Kalman and R. S. Bucy, "New Results in Linear Filtering and Prediction Theory," ASME Trans. (J. of Basic Eng.) vol. 83D, pp. 95-108; March, 1961.

[38] R. E. Kalman, Y. C. Ho and K. S. Narendra, "Controllability of Linear Dynamical Systems," Contrib. Differential Equations, vol. 1, pp. 189-213; 1962.

[39] R. K. Keenan, "Free Oscillations in Periodically Varying Systems," Report no. MEE 65-2, School of Engineering, Monash University, Australia; April, 1965.

[40] R. K. Kinariwala, "Analysis of Time-Varying Networks," IRE Conv. Record, vol. 9, part 4, pp. 268-276; 1961.

[41] J. Klapper and M. Ghausi, "On the Properties of Combinations of Parametric Elements," IEEE Conv. Record, vol. 12, part 1, pp. 357-363; March, 1964.

[42] E. Kreindler and P. E. Sarachik, "On the Concepts of Controllability and Observability of Linear Systems," IEEE Trans. on Automatic Control, vol. AC-9, pp. 129-136; April, 1964.

[43] E. S. Kuh, "Stability of Linear Time-Varying Networks — The State Space Approach," IEEE Trans. on Circuit Theory, vol. CT-12, pp. 150-156; June, 1965.

[44] E. S. Kuh and R. A. Rohrer, "The State-Variable Approach to Network Analysis," Proc. IEEE, vol. 53, pp. 672-686; July, 1965.

[45] J. P. LaSalle, "The Time-Optimal Control Problem," Contributions to Nonlinear Oscillations, Princeton University Press, Princeton, N. J., vol. 5, pp. 1-24; 1960.

[46] H. E. Meadows, "Analysis of Linear Time-Variable RC Networks," IEEE Conv. Record, vol. 11, part 2, pp. 34-42; 1963.

[47] B. S. Morgan, "The Synthesis of Single Variable Systems by State Variable Feedback," Proc. Allerton Conf. on Circuit and System Theory, University of Illinois, Urbana; November, 1963.

[48] R. T. Mohr, "A New Nonreciprocal Transmission Line Device," Proc. IEEE (Correspondence), vol. 52, p. 612; May 1964.

[49] I. H. Mufti, "On the Reduction of a System to Canonical (Phase-Variable) Form," IEEE Trans. on Automatic Control, vol. AC-10, pp. 206-207; April, 1965.

[50] L. S. Pontryagin, "Optimal Control Processes," Uspekhi Matem. Nauk., vol. 14, pp. 3-20; 1959 (English translation in American Math. Soc. Trans., Ser. 2, vol. 18, pp. 321-339; 1961.)

[51] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, The Mathematical Theory of Optimal Processes, Interscience, New York; 1962.

[52] R. J. Schwarz and B. Friedland, Linear Systems, McGraw-Hill Book Co., Inc., New York, N. Y.; 1965.

[53] L. M. Silverman and H. E. Meadows, "Controllability and Time-Variable Unilateral Networks," IEEE Trans. on Circuit Theory, vol. CT-12, pp. 308-313; Sept. 1965.

[54] L. M. Silverman and H. E. Meadows, "Degrees of Controllability in Time-Variable Linear Systems," Proceedings of the National Electronics Conf., vol. 21, pp. 689-693; 1965.

[55] L. M. Silverman, "Transformation of Time-Variable Systems to Canonical (Phase-Variable) Form," IEEE Trans. on Automatic Control; to appear.

[56] R. Sivan, "On Zeroing the Output and Maintaining It Zero," IEEE Trans. On Automatic Control, vol. AC-10, pp. 193-194; April, 1965.

[57] A. R. Stubberud, "Reducibility of Linear Systems," IEEE Trans. on Appl. and Ind., vol. 67, pp. 214-219; July, 1963.

[58] A. R. Stubberud, "A Controllability Criterion for a Class of Linear Systems," IEEE Trans. on Appl. and Ind., vol. 68, pp. 411-413; November, 1964.

[59] L. Weiss and R. E. Kalman, "Contributions to Linear System Theory," Technical Report 64-9, Research Institute for Advanced Study, Maryland; March, 1964.

[60] H. J. Wertz, "On the Numerical of a Recurrent Problem: The Vandermonde Matrix," IEEE Trans. on Automatic Control, vol. AC-9, p. 492; October, 1965.

[61] D. C. Youla, "The Synthesis of Linear Dynamical Systems from Prescribed Weighting Patterns," PIB Report PIB MRI-1271-65, Department of Electrophysics, Polytechnic Institute of Brooklyn; June, 1965.

[62] D. C. Youla and P. Tissi, "n-Port Synthesis via Reactance Extraction— Part 1," IEEE Conv. Record, vol. 14, part 7, pp. 183-205; 1966.

[63] L. A. Zadeh, "Time-Varying Networks, 1," Proc. IRE, vol. 49, No. 10, pp. 1488-1502; Oct., 1961.

[64] L. A. Zadeh and C. A. Desoer, Linear System Theory, McGraw-Hill Book Co., Inc., New York, N. Y.; 1963.

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Columbia University, Dept. of Electrical Engr. New York, New York 10027	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

REPRESENTATION AND REALIZATION OF TIME-VARIABLE LINEAR SYSTEMS

4. DESCRIPTIVE NOTES (Type of report and inclusive dates) Technical Report

5. AUTHOR(S) (Last name, first name, initial)

SILVERMAN, LEONARD M.

6. REPORT DATE June 1, 1966	7a. TOTAL NO. OF PAGES 194	7b. NO. OF REFS 64

8a. CONTRACT OR GRANT NO. NONR 4259(04)	9a. ORIGINATOR'S REPORT NUMBER(S) Technical Report # 94
b. PROJECT NO. NR 044-311	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

10. AVAILABILITY/LIMITATION NOTICES

Qualified requestors may obtain copies of this report from DDC.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Department of the Navy Office of Naval Research Mathematics Branch

13. ABSTRACT

Systems of linear time-variable differential equations are studied, with particular emphasis on identifying those system properties and concepts that can be characterized without knowledge of the equations' solution. Criteria are developed for determining the degree of controllability and observability of such systems. These criteria are based on the rank of matrices formed directly from the system coefficients. A transformational property of these matrices is utilized in a procedure for reducing noncontrollable and non-observable systems to lower dynamic order. Also obtained are new methods for characterizing and generating equivalent system representations, including criteria for equivalence and zero-state time invariance of time-variable systems. Based on the theory of equivalent systems, a new approach to the synthesis of nonstationary impulse response matrices is developed. This method, which does not require an a priori assumption of separability, provides a systematic procedure for realizing a wide class of responses. Application is also made to time-variable electrical networks. By relating the concepts of controllability and unilateral transmission, the existence of a class of unilateral networks composed solely of two-terminal RC (time-variable) components is established.

DD FORM 1 JAN 64 1473

Unclassified

Security Classification

Unclassified
Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
TIME-VARIABLE NETWORKS						
TIME-VARIABLE SYSTEMS						
CONTROLLABILITY						
OBSERVABILITY						
SYSTEM REDUCTION						
EQUIVALENCE						
REPRESENTATION						
REALIZATION						
SYNTHEISS						
UNILATERAL NETWORKS						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through ________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through ________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through ________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.

Unclassified
Security Classification